Mathematics 13 : Elementary Statistics

Unit 2 : Section 5

Lesson 28 :

The Normal Probability Distribution

The Normal Probability Distribution

The normal probability distribution calculates the probability associated with a continuous random variable (X) that exhibits a particular bell-shaped distribution (called the Gaussian distribution).

X

The Normal Probability Distribution

Even in cases where the random variable does not precisely follow the Gaussian distribution, the normal probability distribution can be used to approximate probabilities of random variables that reasonably resemble a bell-shaped distribution.

The Normal Probability Distribution

The normal random variable X is continuous. As such, it can take on any value over a certain interval.

In theory, the value of X can result in an infinite number of digits. In reality, the value of X is restricted or rounded either due to practical considerations or limitations in measurement.

The Normal Probability Distribution

For instance,

if a person's height was measured as 6 feet tall, chances are that person is not exactly 6 feet tall. For that person to be exactly 6 feet tall, their height must equal 6.000000 $\cdots$ feet exactly. It is not possible to measure this result to this degree of accuracy due to limitations and restrictions in the measurement process.

The Normal Probability Distribution

Thus, the probability that a normal random variable X exactly equals a specific value is essentially zero (that is, $p_{X=x} \approx 0$).

Consequently, probabilities involving the normal probability distribution are calculated over intervals (that is, for a range of values from some lower limit to an upper limit).

The Normal Probability Distribution

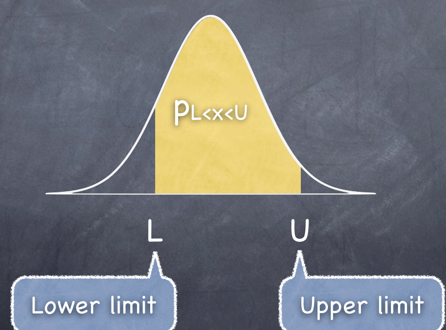
The probability that a normal random variable (X) takes on any value within the interval from some lower limit (L) to some upper limit (U) is given by

$$p_{L<X<U} = \int_{L}^{U} \frac{1}{\sigma_X \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} dx$$

This formula is programmed into the TI-84 calculator.

The Normal Probability Distribution

The normalcdf(L,U,μ_x,σ_x) command calculates the probability of $p_{L<x<U}$.



The Normal Probability Distribution

For instance,

the heights of adult males are normally distributed with a mean of 69.42 inches and a standard deviation of 3.15 inches.

Find the probability that a randomly selected adult male will be between 60 inches (5 feet) and 72 inches (6 feet) tall.

The Normal Probability Distribution

Normal with $\mu_x = 69.42$ and $\sigma_x = 3.15$

$p_{60<x<72}$ = normalcdf(60,72,69.42,3.15)

≈ 0.792

Thus, the probability that a randomly selected adult male will be between 60 inches (5 feet) and 72 inches (6 feet) tall is 79.2%.

The Normal Probability Distribution

The normalcdf(L,U,μ_x,σ_x) command on the TI-84 calculator calculates the probability that a normal random variable X with mean μ_x and standard deviation σ_x takes on any value within the interval from L to U.

When L, U, μx, and σx are given to the normalcdf command,

normalcdf(L,U,μ_x,σ_x) = $p_{L<X<U}$

the probability that X falls between L and U is calculated.

The Normal Probability Distribution

The normalcdf command finds a particular normal probability when given certain values of the random variable X.

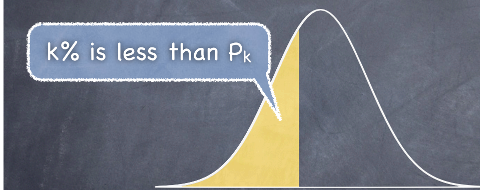
$$\text{normalcdf}(L,U,\mu_x,\sigma_x) = p_{L<X<U}$$

$$P_k = \text{invNorm}(k/100,\mu_x,\sigma_x)$$

The invNorm command finds a particular value of the random variable X when given a certain normal probability.

The Normal Probability Distribution

The invNorm($k/100,\mu_x,\sigma_x$) command calculates the k^{th} percentile P_k.



$$P_k = \text{invNorm}(k/100,\mu_x,\sigma_x)$$

The k^{th} percentile

The Normal Probability Distribution

For instance,

the heights of adult males are normally distributed with a mean of 69.42 inches and a standard deviation of 3.15 inches.

Find the 75^{th} percentile for the heights of adult males.

The Normal Probability Distribution

Normal with μ_x = 69.42
and σ_x = 3.15

$$\text{invNorm}(75/100, 69.42, 3.15) = P_{75}$$

$$\approx 71.5$$

Thus, the 75^{th} percentile for the heights of adult males is 71.5 inches.

The Normal Probability Distribution

In statistics, the normal probability distribution is relied on considerably to both determine precisely and approximately probabilities of random variables that display a bell-shaped distribution.

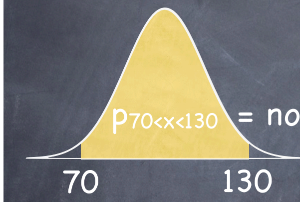
The normal probability distribution also provides the basis for several important statistical analysis procedures.

Example 1

IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. What percentage of people in the population have an IQ score between 70 and 130?

Example 1

Normal with $\mu_x = 100$ and $\sigma_x = 15$



= normalcdf(70,130,100,15)

≈ 0.954

About 95.4% of people in the population have an IQ score between 70 and 130.

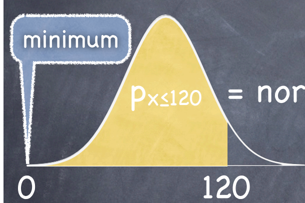
Example 2

IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. What percentage of people in the population have an IQ score of at most 120?

Example 2

Normal with $\mu_x = 100$ and $\sigma_x = 15$

If the minimum value is not known, enter -9E99 (-∞) instead.



= normalcdf(0,120,100,15)

≈ 0.909

About 90.9% of people in the population have an IQ score of at most 120.

Example 3

IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. What percentage of people in the population have an IQ score above 90?

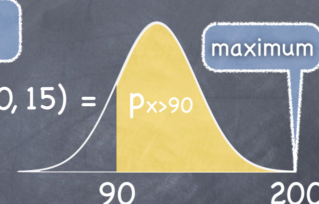
Example 3

Normal with $\mu_x = 100$ and $\sigma_x = 15$

If the maximum value is not known, enter 9E99 (∞) instead.

normalcdf(90, 200, 100, 15) =

≈ 0.748



About 74.8% of people in the population have an IQ score above 90.

Example 4

Membership in Mensa, the high IQ society, is open to persons who have attained an IQ score within the upper two percent of the general population.

If IQ scores are normally distributed with a mean of 100 and a standard deviation of 15, which IQ scores would qualify a person for membership in Mensa?

Example 4

Normal with $\mu_x = 100$ and $\sigma_x = 15$

P_{98} = invNorm(98/100,100,15)

≈ 130.8

Thus, IQ scores of 131 or higher would qualify a person for membership in Mensa.

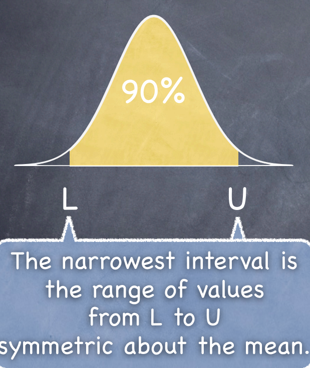
Example 5

IQ scores are normally distributed with a mean of 100 and a standard deviation of 15.

Find the narrowest interval for IQ scores such that 90% of the people in the population have an IQ score that falls within it.

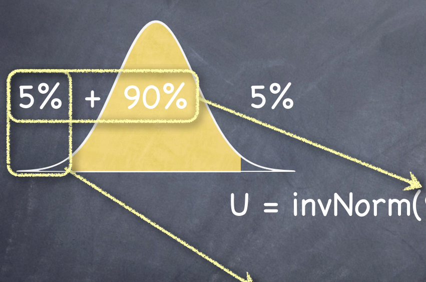
Example 5

Normal with $\mu_x = 100$ and $\sigma_x = 15$



Example 5

Normal with $\mu_x = 100$ and $\sigma_x = 15$



U = invNorm(95/100,100, 15)

L = invNorm(5/100,100, 15)

Example 5

Normal with $\mu_x = 100$ and $\sigma_x = 15$

U = invNorm(95/100,100, 15) ≈ 124.7

L = invNorm(5/100,100, 15) ≈ 75.3

Thus, the narrowest interval for IQ scores such that 90% of the people in the population have an IQ score that falls within it ranges from a low of about 75 to a high of around 125.

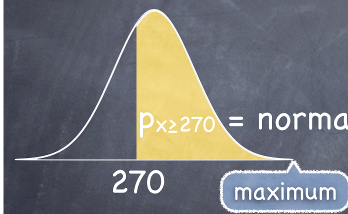
Example 6

The duration of human pregnancy follows a normal distribution with a mean of 274.5 days and a standard deviation of 13.6 days.

Find the probability that a human pregnancy will last at least 270 days.

Example 6

Normal with $\mu_x = 274.5$
and $\sigma_x = 13.6$



$p_{x \geq 270}$ = normalcdf(270,9E99,274.5,13.6)
≈ 0.630

Thus, the probability that a human pregnancy will last at least 270 days is about 63.0%.

Example 7

The duration of human pregnancy follows a normal distribution with a mean of 274.5 days and a standard deviation of 13.6 days.

A preterm (or premature) birth occurs when the duration of the pregnancy is less than 37 weeks. What percentage of births result in a preterm birth?

Example 7

Normal with $\mu_x = 274.5$
and $\sigma_x = 13.6$

$p_{preterm} = p_{x<37 \text{ weeks} \cdot 7 \text{ days/week}} = p_{x<259 \text{ days}}$

= normalcdf(0,259,274.5,13.6)

≈ 0.127

Thus, about 12.7% of births result in a preterm birth.

Example 8

The duration of human pregnancy follows a normal distribution with a mean of 274.5 days and a standard deviation of 13.6 days.

Would it be unusual for a human pregnancy to last anywhere from 300 to 330 days?

An event is considered to be unusual when $p_E \leq 0.05$

Example 8

Normal with $\mu_x = 274.5$
and $\sigma_x = 13.6$

$p_{300 \leq x \leq 330}$

= normalcdf(300,330,274.5,13.6)

$\approx 0.030 \leq 0.05$

Since $p_{300 \leq x \leq 330} \approx 3.0\% \leq 5\%$, it would be unusual for a human pregnancy to last anywhere from 300 to 330 days.

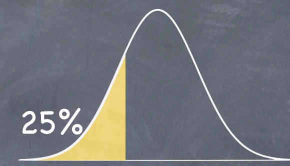
Example 9

The duration of human pregnancy follows a normal distribution with a mean of 274.5 days and a standard deviation of 13.6 days.

Find the first quartile for the duration of human pregnancies and interpret the result.

Example 9

Normal with $\mu_x = 274.5$ and $\sigma_x = 13.6$



$\text{invNorm}(0.25, 274.5, 13.6) = Q_1 = P_{25}$

≈ 265.3

One quarter (or 25%) of human pregnancies have a duration of less than 265 days.

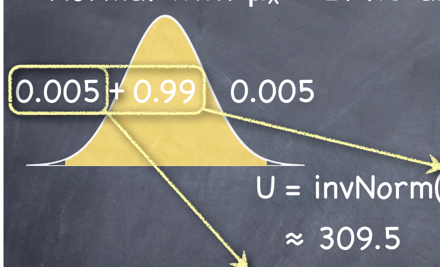
Example 10

The duration of human pregnancy follows a normal distribution with a mean of 274.5 days and a standard deviation of 13.6 days.

Find the narrowest interval for the duration of human pregnancy such that 99% of all pregnancies fall between.

Example 10

Normal with $\mu_x = 274.5$ and $\sigma_x = 13.6$



$U = \text{invNorm}(0.995, 274.5, 13.6)$

≈ 309.5

$L = \text{invNorm}(0.005, 274.5, 13.6)$

≈ 239.5

Example 10

Thus, the narrowest interval for the duration of human pregnancy such that 99% of all pregnancies fall between ranges from a low of about 239 days to a high of around 310 days.

Your solutions should be clear, complete, and sufficiently detailed in order to demonstrate your understanding and communicate your reasoning and method of solving the problem.

Exercise 1

The weights of adult male Rhesus monkeys (*Macaca mulatta*) are normally distributed with a mean of 7.71 kilograms and a standard deviation of 1.68 kilograms.

(a) What is the probability that a randomly selected adult male Rhesus monkey weighs somewhere between 6 and 7 kilograms?

(b) Find the 67^{th} percentile for the weights of adult male Rhesus monkeys.

Exercise 2

The weights of adult male Rhesus monkeys (*Macaca mulatta*) are normally distributed with a mean of 7.71 kilograms and a standard deviation of 1.68 kilograms.

(a) What proportion of adult male Rhesus monkeys weigh at most 9 kilograms?

(b) What weights would be considered unusually low for adult male Rhesus monkeys?

Exercise 3

The lengths of California Kingsnakes (*Lampropeltis getula californiae*) are normally distributed with a mean of 42.8 inches and a standard deviation of 9.7 inches.

(a) What percentage of California Kingsnakes measure less than 24 inches in length?

(b) Would it be unusual to find a California Kingsnake that is over four feet long?

Exercise 4

The lengths of California Kingsnakes (*Lampropeltis getula californiae*) are normally distributed with a mean of 42.8 inches and a standard deviation of 9.7 inches.

Find the narrowest interval for the lengths of California Kingsnakes such that 95% of all California Kingsnakes fall between.

Exercise 5

According to a study conducted at Purdue University, college students sleep an average of 7.65 hours per night with a standard deviation of 0.98 hours per night. These nightly sleep times of college students displayed an approximate bell-shaped distribution.

(a) Based on these results, what percentage of college students get at least 8 hours of sleep per night?

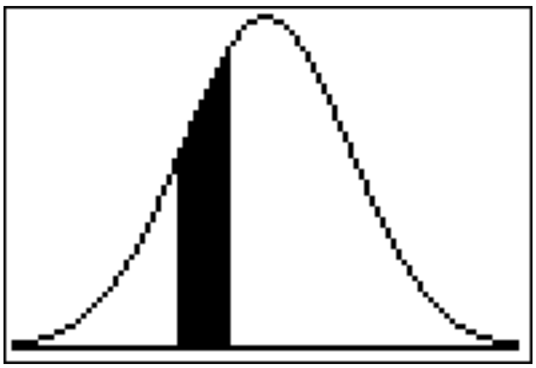
(b) Based on these results, would it be unusual for a college student to get under 6 hours of sleep in a night?

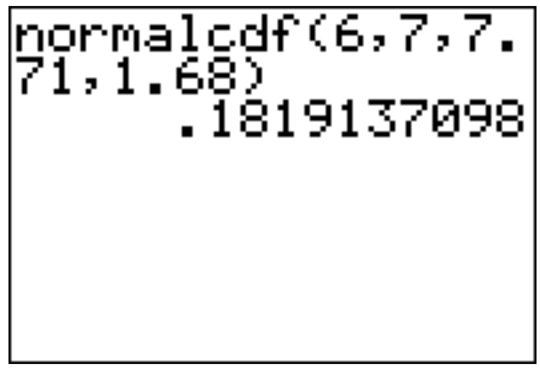
> Your solutions should be clear, complete, and sufficiently detailed in order to demonstrate your understanding and communicate your reasoning and method of solving the problem.

Exercise 1

The weights of adult male Rhesus monkeys (*Macaca mulatta*) are normally distributed with a mean of 7.71 kilograms and a standard deviation of 1.68 kilograms.

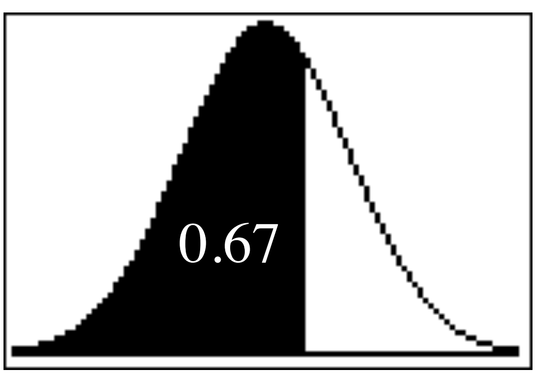
(a) What is the probability that a randomly selected adult male Rhesus monkey weighs somewhere between 6 and 7 kilograms?

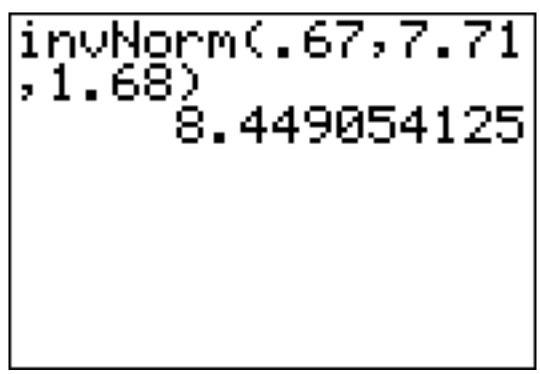




$$p_{\text{between 6 and 7}} = p_{6<X<7} \approx 0.182$$

(b) Find the 67th percentile for the weights of adult male Rhesus monkeys.



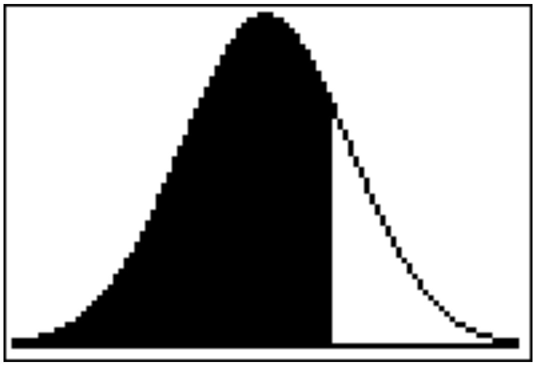


$$P_{67} \approx 8.45\ kg$$

Exercise 2

The weights of adult male Rhesus monkeys (*Macaca mulatta*) are normally distributed with a mean of 7.71 kilograms and a standard deviation of 1.68 kilograms.

(a) What proportion of adult male Rhesus monkeys weigh at most 9 kilograms?

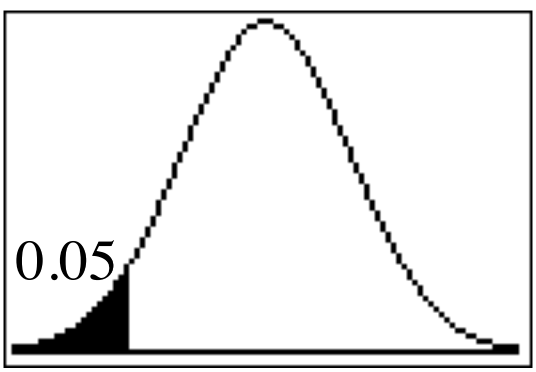


```
normalcdf(0,9,7.
71,1.68)
           .778711814
```

$$p_{\text{at most 9}} = p_{X \le 9} \approx 0.779$$

Thus, the proportion of adult male Rhesus monkeys that weigh at most 9 kg is about 0.779.

(b) What weights would be considered unusually low for adult male Rhesus monkeys?




```
invNorm(.05,7.71
,1.68)
          4.946645908
```

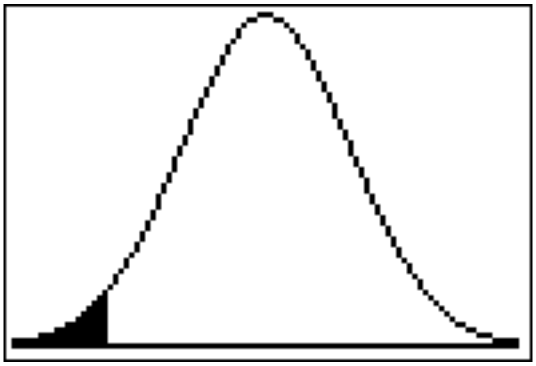
$$P_5 \approx 4.95\ kg$$

Thus, 4.95 kg or less would be considered an unusually low weight for adult male Rhesus monkeys.

Exercise 3

The lengths of California Kingsnakes (*Lampropeltis getula californiae*) are normally distributed with a mean of 42.8 inches and a standard deviation of 9.7 inches.

(a) What percentage of California Kingsnakes measure less than 24 inches in length?



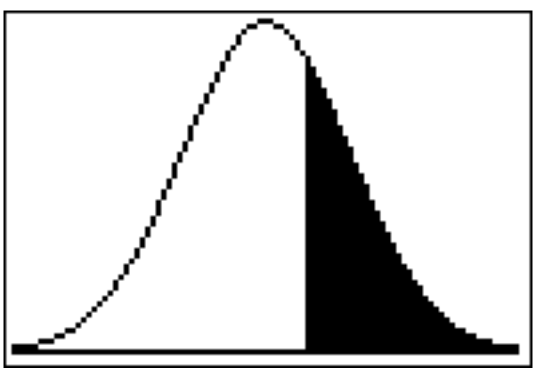
```
normalcdf(0,24,4
2.8,9.7)
          .0262976224
```

$$p_{\text{less than 24}} = p_{X<24} \approx 2.6\%$$

Thus, 2.6% of California Kingsnakes measure less than 24 inches in length.

(b) Would it be unusual to find a California Kingsnake that is over four feet long?

$$4 \; feet = 48 \; inches$$



```
normalcdf(48,9E9
9,42.8,9.7)
          .2959507506
```

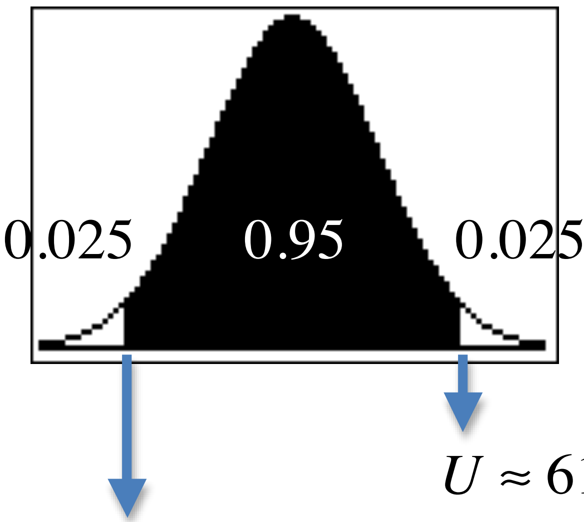
$$p_{\text{over 4 feet}} = p_{X>48} \approx 29.6\%$$

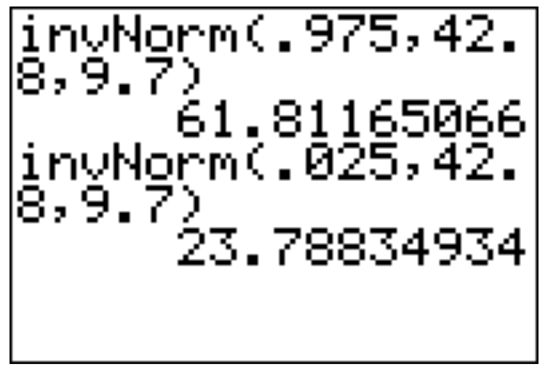
No. Since the probability of finding a California Kingsnake that is over four feet long, 29.6%, is not 5% or less, it would not be unusual to find a California Kingsnake that is over four feet long.

Exercise 4

The lengths of California Kingsnakes (*Lampropeltis getula californiae*) are normally distributed with a mean of 42.8 inches and a standard deviation of 9.7 inches.

Find the narrowest interval for the lengths of California Kingsnakes such that 95% of all California Kingsnakes fall between.





$$U \approx 61.8 \; inches$$

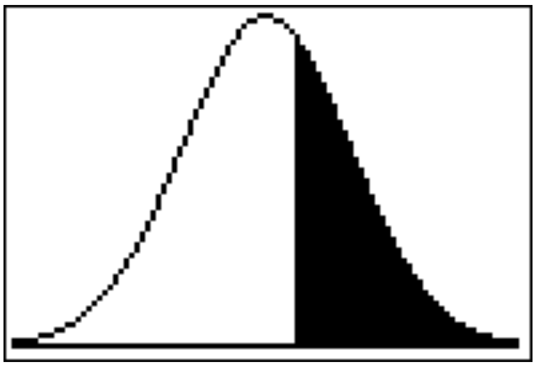
$$L \approx 23.8 \; inches$$

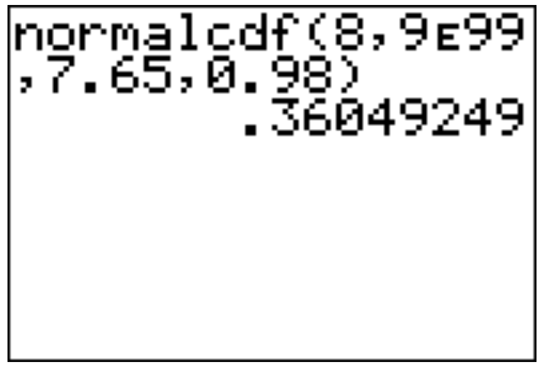
Thus, the narrowest interval for the lengths of California Kingsnakes such that 95% of all California Kingsnakes fall between ranges from a low of 23.8 inches to a high of 61.8 inches.

Exercise 5

According to a study conducted at Purdue University, college students sleep an average of 7.65 hours per night with a standard deviation of 0.98 hours per night. These nightly sleep times of college students displayed an approximate bell-shaped distribution.

(a) Based on these results, what percentage of college students get at least 8 hours of sleep per night?

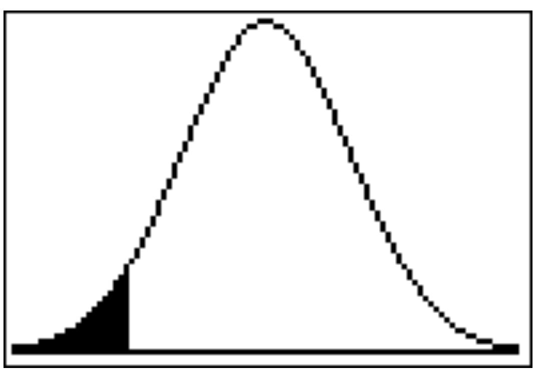


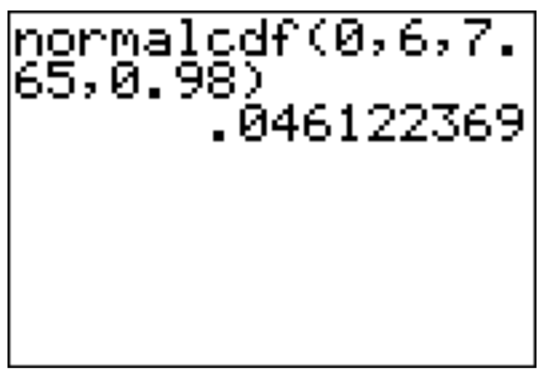


$$p_{\text{at least 8}} = p_{X \geq 8} \approx 36.0\%$$

Based on these results, 36.0% of college students get at least 8 hours of sleep per night.

(b) Based on these results, would it be unusual for a college student to get under 6 hours of sleep in a night?





$$p_{\text{under 6}} = p_{X<6} \approx 4.6\%$$

Yes. Since the probability that a college student gets under 6 hours of sleep in a night, 4.6%, is 5% or less, it would be unusual for a college student to get under 6 hours of sleep in a night.