Mathematics 13 : Elementary Statistics

Unit 2 : Section 4

Lesson 27 :

The Binomial Probability Distribution

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a certain amount (x) of one possible outcome out of a total number (n) of attempts when each independent attempt can result in only one (with probability p) of two possible outcomes.

The Binomial Probability Distribution

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An event occurs or Does not occur

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a certain amount (x) of one possible outcome out of a total number (n) of attempts when each independent attempt can result in only one (with probability p) of two possible outcomes.

The result of one attempt does not affect the result of the other attempts.

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a certain amount (x) of one possible outcome out of a total number (n) of attempts when each independent attempt can result in only one (with probability p) of two possible outcomes.

In order to calculate probabilities using the binomial probability distribution, the values of the three variables n, p, and x must be determined.

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a certain amount (x) of one possible outcome out of a **total number (n) of attempts** when each independent attempt can result in only one (with probability p) of two possible outcomes.

> represents the total number of times the binomial random process is being repeated

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a certain amount (x) of one possible outcome out of a total number (n) of attempts when each independent attempt can result in only one (with **probability p**) of two possible outcomes.

> represents the probability that one particular outcome occurs each time the binomial random process is conducted

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a **certain amount (x) of one possible outcome** out of a total number (n) of attempts when each independent attempt can result in only one (with probability p) of two possible outcomes.

> represents the value of the binomial random variable for which the probability is being calculated

The Binomial Probability Distribution

The binomial probability distribution calculates the probability of getting a **certain amount (x) of one possible outcome** out of a total number (n) of attempts when each independent attempt can result in only one (with probability p) of two possible outcomes.

> The binomial random variable X is discrete. The possible values of X are $x = 0, 1, 2, \cdots, \text{or } n$

The Binomial Probability Distribution

The binomial probability distribution formula is given by

$$p_x = \frac{n!}{x!\,(n-x)!} \cdot p^x \cdot (1-p)^{n-x}$$

This formula is programmed into the TI-84 calculator.

The Binomial Probability Distribution

The **binompdf(n,p,x)** command calculates the probability of p_x.

> The values of n, p, and x are entered into the binomial probability distribution formula to calculate the probability of equaling the entered x value only.

The **binomcdf(n,p,x)** command calculates the probability of $p_{X \le x} = p_0 + p_1 + p_2 + \ldots + p_x$.

> The values of n, p, and X = 0, 1, 2, ..., x are entered into the binomial probability distribution formula and the resulting probabilities are summed (cumulated).

The Binomial Probability Distribution

For instance,

if a fair coin is tossed 10 times, the probability that exactly 5 of those 10 tosses results in heads is

binompdf(10,1/2,5) ≈ 0.246

n = 10 p = 1/2 x = 5

This follows a binomial random process since tossing a coin can result in only one of two possible outcomes (heads or tails).

The Binomial Probability Distribution

For instance,

if a fair coin is tossed 10 times, the probability that 5 or fewer of those 10 tosses results in heads is

binomcdf(10,1/2,5) ≈ 0.623

n = 10 p = 1/2 x ≤ 5

This follows a binomial random process since tossing a coin can result in only one of two possible outcomes (heads or tails).

The Binomial Probability Distribution

The mean for the binomial random variable X is given by

$$\mu_X = n \cdot p$$

The standard deviation for the binomial random variable X is given by

$$\sigma_X = \sqrt{n \cdot p \cdot (1-p)}$$

The Binomial Probability Distribution

For instance,

calculate the mean and the standard deviation for the random variable X which represents the number of heads observed when a fair coin is tossed 10 times.

$$\mu_X = n \cdot p = 10 \cdot 1/2 = 5$$

$$\sigma_X = \sqrt{n \cdot p \cdot (1-p)}$$
$$= \sqrt{10 \cdot 1/2 \cdot (1-1/2)} \approx 1.6$$

The Binomial Probability Distribution

The binomial probability distribution naturally arises in numerous situations in the application of probability and statistical methods.

Learning to recognize these situations where the binomial probability distribution can be applied will produce more efficient and effective results.

Example 1

A careful examination of birth records over the past 70 years has revealed that there are 105 boys born for every 100 girls that are born.

If a couple plans on having three children, what is the probability that they will have exactly two boys?

Example 1

$p_{x=2}$ = binompdf(3,105/205,2) $\approx$ 0.384

Assuming that the probability of a boy being born is 105/205 (51.2%), if a couple plans on having three children, the probability that they will have exactly two boys is 38.4%.

Example 2

A careful examination of birth records over the past 70 years has revealed that there are 105 boys born for every 100 girls that are born.

If a couple plans on having three children, what is the chance that less than half of them are boys?

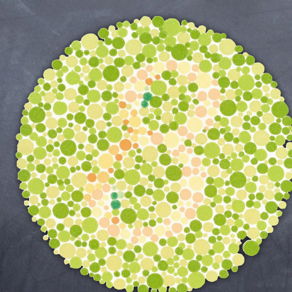
Example 2

$p_{x\leq 1}$ = binomcdf(3,105/205,1) $\approx$ 0.482

Assuming that the probability of a boy being born is 105/205 (51.2%), if a couple plans on having three children, there is a 48.2% chance that less than half of them are boys.

Example 3

What do you see when you look at the following image?



Example 3

Color blindness (or color vision deficiency) is the inability of an individual to distinguish the differences between certain colors. This condition results from the absence of color sensing pigments in the cone cells of the retina.

Color blindness is a sex linked trait. The genes that code for these color sensing pigments are located on the X chromosome.

Example 3

Individuals that are not color blind possess a working form of these photo sensing genes (denoted by X).

When a mutation occurs in these genes, the color sensing pigments can not be produced (denoted by x).

Example 3

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, find the probability that they would have no color blind children.

Binomial with $n = 3$, $p = 0.25$, and $x = 0$

	X	Y
X	XX	XY
x	xX	xY

$p = ¼ = 0.25 = 25\%$

Example 3

$p_{x=0}$ = binompdf(3,0.25,0) ≈ 0.422

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, the probability that they would have no color blind children is 42.2%.

Example 4

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, find the probability that they would have at most one color blind child.

Binomial with $n = 3$, $p = 0.25$, and $x \leq 1$

Example 4

$p_{x\leq 1}$ = binomcdf(3,0.25,1) ≈ 0.844

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, the probability that they would have at most one color blind child is 84.4%.

Example 5

A non-color blind father (XY) and a carrier mother (Xx) have a 25% chance of having a color blind child each time they have a child.

If this couple were to have three children, would it be considered unusual if all of their children were color blind?

Example 5

Binomial with $n = 3$, $p = 0.25$, and $x = 3$

$p_{all} = p_{x=3}$ = binompdf(3,0.25,3) ≈ 0.016

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, it would be considered unusual if all three of their children were color blind since $p_{all} \approx 0.016 \leq 0.05$.

Example 6

A non-color blind father (XY) and a carrier mother (Xx) have a 25% chance of having a color blind child each time they have a child.

How many color blind children could this couple expect to have if they were to have three children total?

Example 6

$$E(X) = \mu_X = n \cdot p = 3 \cdot 0.25 = 0.75$$

If a non-color blind father (XY) and a carrier mother (Xx) were to have three children, they could expect to have about one color blind child.

Example 7

If someone were to bet $1 on red in the game of roulette 40 separate times, what is the chance that they would win half of those bets?

Example 7

$p_{half} = p_{x=20}$

$= \text{binompdf}(40,18/38,20) \approx 0.119$

If someone were to bet $1 on red in the game of roulette 40 separate times, there is an 11.9% chance that they would win half of those bets.

Example 8

If someone were to bet $1 on red in the game of roulette 40 separate times, what is the chance that they would end up with a profit?

To end up with a profit, they must win more money than they lose. Since betting on red in the game of roulette is an even money bet, they must win more than half of these bets (X > 20).

Example 8

$p_{profit} = p_{x>20} = 1 - p_{x\le 20}$

$= 1 - \text{binomcdf}(40,18/38,20) \approx 0.311$

If someone were to bet $1 on red in the game of roulette 40 separate times, there is a 31.1% chance that they would end up with a profit.

Example 9

According to Snapple Real Fact #52, 11% of people in the world are left-handed.

In a classroom containing 32 students, find the probability that only one of them is left-handed.

Example 9

Binomial with $n = 32$, $p = 0.11$, and $x = 1$

p_1 = binompdf(32,0.11,1) ≈ 0.095

In a classroom containing 32 students, the probability that only one of them is left-handed is 9.5%.

Example 10

According to Snapple Real Fact #52, 11% of people in the world are left-handed.

In a classroom containing 32 students, find the probability that less than three of them are left-handed.

Example 10

Binomial with $n = 32$, $p = 0.11$,
and $x =$ 0, 1, or 2

$p_{x<3} = p_{x\leq 2}$ = binomcdf(32,0.11,2) ≈ 0.301

In a classroom containing 32 students, the probability that less than three of them are left-handed is 30.1%.

Example 11

According to Snapple Real Fact #52, 11% of people in the world are left-handed.

In a classroom containing 32 students, find the probability that anywhere from two to four of them are left-handed.

Example 11

Binomial with $n = 32$, $p = 0.11$,
and $x =$ 2, 3, or 4

$p_{x=2,\,3,\text{ or }4} = p_2 + p_3 + p_4$

= binomcdf(32,0.11,4) – binomcdf(32,0.11,1)

≈ 0.608

In a classroom containing 32 students, the probability that anywhere from two to four of them are left-handed is 60.8%.

Example 12

According to Snapple Real Fact #52, 11% of people in the world are left-handed.

In a classroom containing 32 students, find the probability that the number of left-handed students falls within two standard deviations of the mean.

Example 12

Binomial with n = 32 and p = 0.11

$$\mu_X = n \cdot p = 32 \cdot 0.11 = 3.52$$

$$\sigma_X = \sqrt{n \cdot p \cdot (1-p)} = \sqrt{32 \cdot 0.11 \cdot (1-0.11)} \approx 1.77$$

$-0.02 = 3.52 - 2 \cdot 1.77$ $\qquad$ $3.52 + 2 \cdot 1.77 = 7.06$

Example 12

Binomial with n = 32, p = 0.11, and x = 0, 1, 2, 3, 4, 5, 6, or 7

$p_{x \le 7}$ = binomcdf(32,0.11,7)

≈ 0.980

In a classroom containing 32 students, the probability that the number of left-handed students falls within two standard deviations of the mean is 98.0%.

Your solutions should be clear, complete, and sufficiently detailed in order to demonstrate your understanding and communicate your reasoning and method of solving the problem.

Exercise 1

A zoologist has determined that Loggerhead Sea Turtle eggs hatch about 65.2% of the time. What is the probability that exactly 70 out of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch?

Exercise 2

A zoologist has determined that Loggerhead Sea Turtle eggs hatch about 65.2% of the time. What is the chance that 66 or fewer of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch?

Exercise 3

The results of a nationwide Gallup poll revealed that 59.3% of people polled were in favor of the death penalty. Use the results of this Gallup poll to estimate the probability that a majority of the 12 members of a randomly selected jury would be in favor of the death penalty.

Exercise 4

The results of a nationwide Gallup poll revealed that 59.3% of people polled were in favor of the death penalty. Based on the results of this Gallup poll, would it be unusual if all 12 members of a randomly selected jury were in favor of the death penalty?

Exercise 5

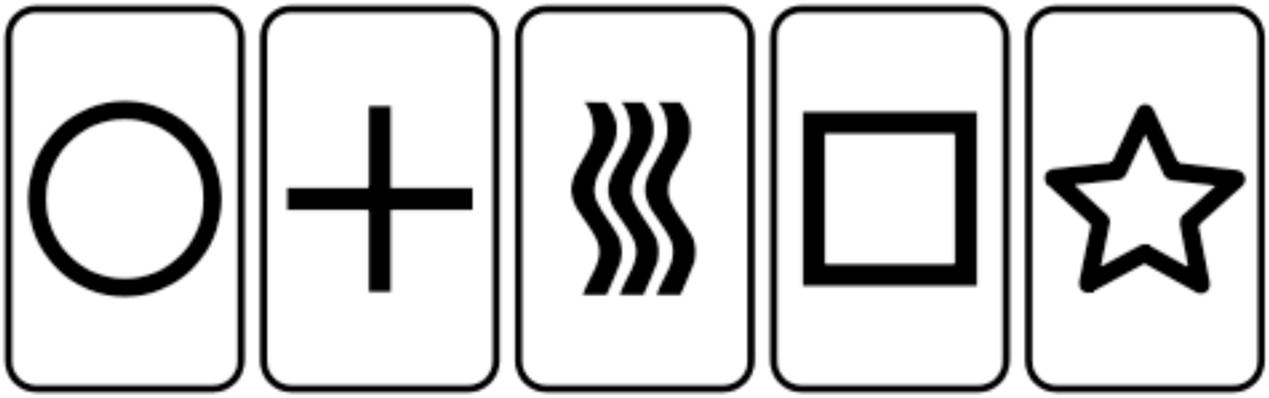
If someone were to bet $1 on the second dozen numbers in the game of roulette 33 separate times, how many of these bets should they expect to win?

Exercise 6

If someone were to bet $1 on the second dozen numbers (which pays 2 to 1) in the game of roulette 33 separate times, what is the chance that they would end up losing money?

Exercise 7

A psychologist has conducted a series of experiments to scientifically study the phenomenon known as extrasensory perception (or ESP). In one experiment, a test subject is brought into a room and seated in a chair facing a blank wall. The experimenter stands behind the seated test subject and randomly selects one of the following five cards.



The experimenter then asks the test subject to identify the symbol on the randomly selected card. The test subject's response is recorded along with the symbol on the randomly selected card. This process is repeated a total of 25 times with the same test subject. Before the test subject gives their consent to taking part in the experiment, the experimenter explains the process of the experiment and shows the symbols on the cards to the test subject.

If the test subject does not possess the power of ESP and is merely guessing the symbol of the randomly selected card, would it be unusual to observe the test subject correctly identifying the symbol nine or more times in this experiment?

Your solutions should be clear, complete, and sufficiently detailed in order to demonstrate your understanding and communicate your reasoning and method of solving the problem.

Exercise 1

A zoologist has determined that Loggerhead Sea Turtle eggs hatch about 65.2% of the time. What is the probability that exactly 70 out of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch?

The binomial probability distribution with n = 107, p = 0.652, and x = 70 can be used since a Loggerhead Sea Turtle egg either hatches or it does not (two possible outcomes).

```
binompdf(107,0.6
52,70)
      .0807786377
```

$$p_{\text{exactly 70}} = p_{70}$$

The probability that exactly 70 out of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch is about 8.1%.

Exercise 2

A zoologist has determined that Loggerhead Sea Turtle eggs hatch about 65.2% of the time. What is the chance that 66 or fewer of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch?

The binomial probability distribution with n = 107, p = 0.652, and $x \le 66$ can be used since a Loggerhead Sea Turtle egg either hatches or it does not (two possible outcomes).

```
binomcdf(107,0.6
52,66)
      .2520781412
```

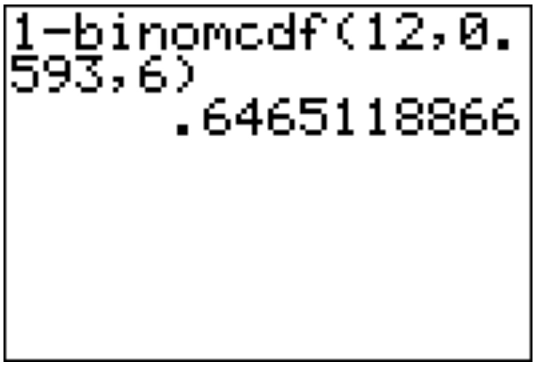
$$p_{\text{66 or fewer}} = p_{X \le 66}$$

The chance that 66 or fewer of a total of 107 Loggerhead Sea Turtle eggs in a nest hatch is about 25.2%.

Exercise 3

The results of a nationwide Gallup poll revealed that 59.3% of people polled were in favor of the death penalty. Use the results of this Gallup poll to estimate the probability that a majority of the 12 members of a randomly selected jury would be in favor of the death penalty.

The binomial probability distribution with n = 12, p = 0.593, and x > 6 can be used since a person is either in favor of the death penalty or they are not (two possible outcomes).



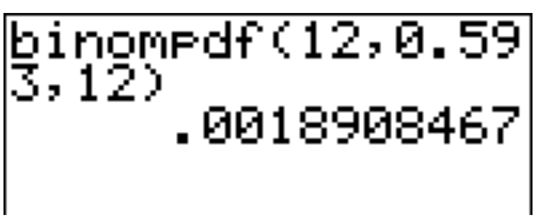
$$p_{\text{majority}} = p_{\text{more than half}} = p_{X>6} = 1 - p_{X\leq 6}$$

The probability that a majority (more than half) of the 12 members of a randomly selected jury would be in favor of the death penalty is 64.7%.

Exercise 4

The results of a nationwide Gallup poll revealed that 59.3% of people polled were in favor of the death penalty. Based on the results of this Gallup poll, would it be unusual if all 12 members of a randomly selected jury were in favor of the death penalty?

The binomial probability distribution with n = 12, p = 0.593, and x = 12 can be used since a person is either in favor of the death penalty or they are not (two possible outcomes).



Yes. Based on the results of this Gallup poll, it would be unusual if all 12 members of a randomly selected jury were in favor of the death penalty since $p_{all} = p_{12} \approx 0.002 \leq 0.05$.

Exercise 5

If someone were to bet $1 on the second dozen numbers in the game of roulette 33 separate times, how many of these bets should they expect to win?

The binomial probability distribution with n = 33 and p = 12/38 can be used since someone will either win their bet on the second dozen numbers in the game of roulette or they will not (two possible outcomes).

$$E(X) = \mu_X = n \cdot p = 33 \cdot \frac{12}{38} \approx 10.4$$

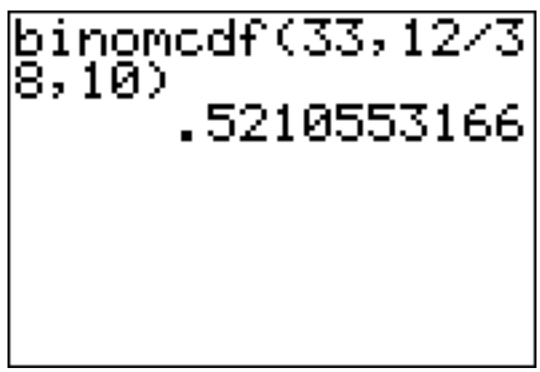
Thus, if someone were to bet $1 on the second dozen numbers in the game of roulette 33 separate times, they should expect to win about 10 or 11 of these bets.

Exercise 6

If someone were to bet $1 on the second dozen numbers (which pays 2 to 1) in the game of roulette 33 separate times, what is the chance that they would end up losing money?

Someone would end up losing money on a bet that pays 2 to 1 odds if they were to win less than one (1) out of every three (2 + 1 = 3) bets that they place. So, when this bet is placed 33 separate times,

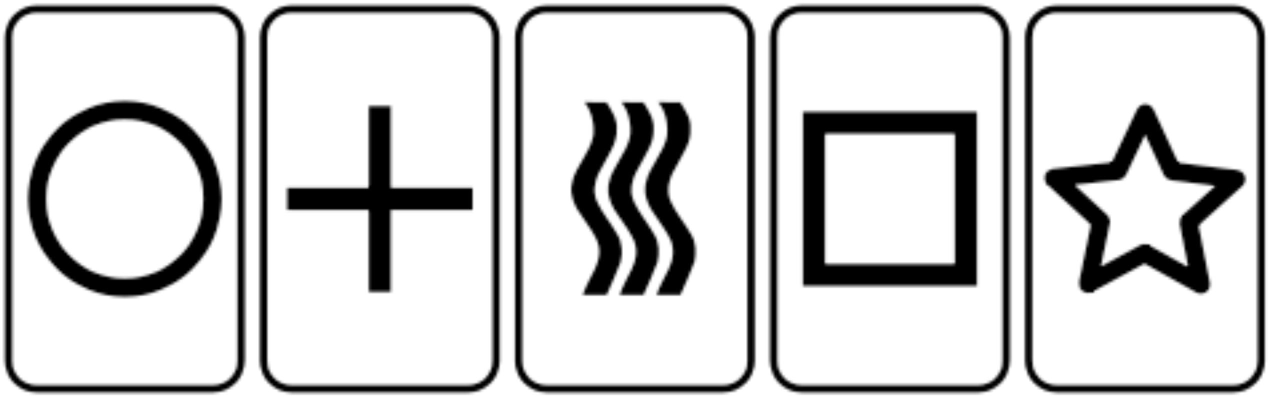
$$\begin{aligned} p_{\text{lose money}} &= p_{X < 33 \cdot \frac{1}{2+1}} \\ &= p_{X<11} \\ &= p_{X \le 10} \end{aligned}$$



Thus, if someone were to bet $1 on the second dozen numbers in the game of roulette 33 separate times, there is a 52.1% chance that they would end up losing money.

Exercise 7

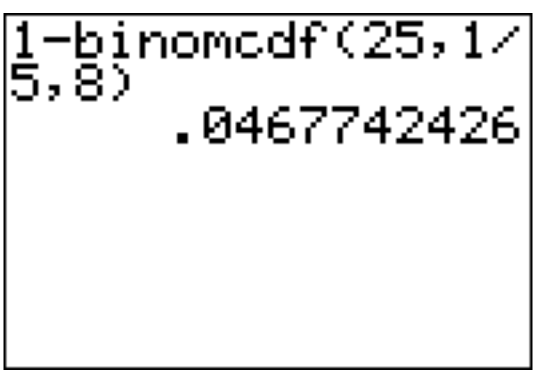
A psychologist has conducted a series of experiments to scientifically study the phenomenon known as extrasensory perception (or ESP). In one experiment, a test subject is brought into a room and seated in a chair facing a blank wall. The experimenter stands behind the seated test subject and randomly selects one of the following five cards.



The experimenter then asks the test subject to identify the symbol on the randomly selected card. The test subject's response is recorded along with the symbol on the randomly selected card. This process is repeated a total of 25 times with the same test subject. Before the test subject gives their consent to taking part in the experiment, the experimenter explains the process of the experiment and shows the symbols on the cards to the test subject.

If the test subject does not possess the power of ESP and is merely guessing the symbol of the randomly selected card, would it be unusual to observe the test subject correctly identifying the symbol nine or more times in this experiment?

The binomial probability distribution with n = 25, p = 1/5, and x ≥ 9 can be used since the test subject will either correctly identify the randomly selected symbol or they will not (two possible outcomes).



$$\begin{aligned} p_{9 \text{ or more}} &= p_{X \geq 9} \\ &= 1 - p_{X<9} \\ &= 1 - p_{X \leq 8} \end{aligned}$$

Yes. Since $p_{9 \text{ or more}} \approx 0.047 \leq 0.05$, it would be unusual to observe the test subject correctly identifying the symbol nine or more times in this experiment if they were merely guessing the symbol.