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Math 31 Summer 2019

7.4 #39-53 odd and 60-62
James Stewart Calculus, 8th edition

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Note: This is not the only way to do these problems, they are merely one way of doing them.

39.

$$\int \left(\frac{1}{x\sqrt{x-1}}\right) dx$$

$$\text{Let } u = \sqrt{x-1}$$
$$u^2 + 1 = x$$
$$2udu = dx$$

$$= \int \left(\frac{2u}{(u^2+1)u}\right) du = \int \left(\frac{2}{u^2+1}\right) du = 2\arctan u + C$$
$$= 2\arctan\sqrt{x-1} + C$$

41.

$$\int \left(\frac{1}{x^2 + x\sqrt{x}}\right) dx$$

$$\text{Let } u = \sqrt{x}$$
$$u^2 = x$$
$$2udu = dx$$

$$= \int \left(\frac{2u}{u^4 + u^2 \cdot u}\right) du = \int \left(\frac{2}{u^3 + u^2}\right) du = \int \left(\frac{2}{u^2(u+1)}\right) du$$

<u>Aside:</u>

$$\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$2 = Au(u+1) + B(u+1) + Cu^2$$

Let $u = 0$

$$2 = B(0+1) \rightarrow B = 2$$

Let $u = -1$

$$2 = C(-1)^2 \rightarrow C = 2$$

Equating u^2 **terms**

$$0 = A + C \rightarrow A = -C \rightarrow A = -2$$

So, we have that

$$\int \left(\frac{1}{x^2 + x\sqrt{x}}\right) dx = \int \left(\frac{2}{u^2(u+1)}\right) du = \int \left(\frac{-2}{u} + \frac{2}{u^2} + \frac{2}{u+1}\right) du$$

$$= -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C = 2\ln\left|\frac{u+1}{u}\right| - \frac{2}{u} + C$$

$$\stackrel{1}{=} 2\ln\left(\frac{\sqrt{x}+1}{\sqrt{x}}\right) - \frac{2}{\sqrt{x}} + C$$

1: We are allowed to suppress the absolute values in the logarithm since $\sqrt{x}$ and $\sqrt{x} + 1$ are both non-negative.

43.

$$\int \left(\frac{x^3}{\sqrt[3]{x^2+1}}\right) dx$$

Let $u = \sqrt[3]{x^2+1}$
$u^3 - 1 = x^2$
$3u^2 = 2xdx$
$\frac{3}{2}u^2 du = xdx$

$$= \int \left(\frac{3u^2}{2} \cdot \frac{u^3-1}{u}\right) du = \frac{3}{2}\int (u^4-u)du = \frac{3}{2}(\frac{u^5}{5} - \frac{u^2}{2}) + C$$

$$= \frac{3u^2}{20}(2u^3-5) = \frac{3(2x^2-3)(x^2+1)^{\frac{2}{3}}}{20} + C$$

45.

$$\int \left(\frac{1}{\sqrt{x} - \sqrt[3]{x}}\right) dx$$

Let $u = \sqrt[6]{x}$
$u^6 = x$
$6u^5 du = dx$

$$= \int \left(\frac{6u^5}{u^3-u^2}\right) du = \int \left(\frac{6u^3}{u-1}\right) du$$

$$= \int \left(\frac{6u^3 - 6u^2 + 6u^2 - 6u + 6u - 6 + 6}{u-1}\right) du = \int \left(6u^2 + 6u + 6 + \frac{6}{u-1}\right) du$$

$$= 2u^3 + 3u^2 + 6u + 6\ln|u+1| + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$$

47.

$$\int \left(\frac{e^{2x}}{e^{2x} + 3e^x + 2}\right) dx$$

Let $u = e^x$
$du = e^x dx$

$$= \int \left(\frac{u}{u^2+3u+2}\right) du = \int \left(\frac{u}{(u+2)(u+1)}\right) du$$

Aside:

$$\frac{u}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

$$u = A(u+1) + B(u+2)$$

Let $u = -1$

$$-1 = B(-1+2) \rightarrow B = -1$$

Let $u = -2$

$$-2 = A(-2+1) \rightarrow A = 2$$

So, we have that

$$\int \left(\frac{e^{2x}}{e^{2x} + 3e^x + 2} \right) dx = \int \left(\frac{u}{(u+2)(u+1)} \right) du = \int \left(\frac{2}{u+2} + \frac{-1}{u+1} \right) du$$

$$= 2\ln|u+2| - \ln|u+1| + C \overset{1}{=} \ln\left(\frac{(u+2)^2}{|u+1|} \right) + C \overset{2}{=} \ln\left(\frac{(e^x+2)^2}{e^x+1} \right) + C$$

1: Note we can suppress the absolute values of $u+2$ since we rewrote it as it's square which is now non-negative.

2: We can suppress the absolute values since $e^x + 1$ is positive definite.

49.

$$\int \left(\frac{\sec^2(t)}{\tan^2(t) + 3\tan(t) + 2} \right) dt$$

Let $u = \tan(t)$
$du = \sec^2(t)dt$

$$= \int \left(\frac{1}{u^2 + 3u + 2} \right) du = \int \left(\frac{1}{(u+2)(u+1)} \right) du$$

Aside:

$$\frac{1}{(u+2)(u+1)} = \frac{A}{u+1} + \frac{B}{u+2}$$

$$1 = A(u+2) + B(u+1)$$

Let $u = -2$

$$1 = B(-2+1) \rightarrow B = -1$$

Let $u = -1$

$$1 = A(-1+2) \rightarrow A = 1$$

So, we have that

$$\int \left(\frac{\sec^2(t)}{\tan^2(t) + 3\tan(t) + 2} \right) dt = \int \left(\frac{1}{(u+2)(u+1)} \right) du$$

$$= \int \left(\frac{1}{u+1} + \frac{-1}{u+2}\right) du = \ln|u+1| - \ln|u+2| + C = \ln\left|\frac{u+1}{u+2}\right| + C$$

$$= \ln\left|\frac{\tan(t)+1}{\tan(t)+2}\right| + C$$

51.

$$\int \left(\frac{1}{1+e^x}\right) dx$$

$$\begin{aligned} \text{Let } u &= e^x \\ du &= e^x dx \\ du &\stackrel{1}{=} u dx \\ \tfrac{1}{u} du &= dx \end{aligned}$$

1: Notice that since we let $u = e^x$ we may say that $e^x dx = u dx$.

$$= \int \left(\frac{1}{u(1+u)}\right) du$$

<u>Aside:</u>

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

$$1 = A(u+1) + Bu$$

Let $u = -1$

$$1 = B(-1) \to B = -1$$

Let $u = 0$

$$1 = A(0+1) \to A = 1$$

So, we have that

$$\int \left(\frac{1}{1+e^x}\right) dx = \int \left(\frac{1}{u(1+u)}\right) du = \int \left(\frac{1}{u} + \frac{-1}{u+1}\right) du$$

$$= \ln|u| - \ln|u+1| + C \stackrel{1}{=} \ln(e^x) - \ln(e^x+1) + C = x - \ln(e^x+1) + C$$

1:We may suppress the absolute values since e^x and $e^x + 1$ are both non-negative.

53.

$$\int (\ln(x^2 - x + 2))dx$$

We shall proceed by integration by parts.

$$u = \ln(x^2 - x + 2) \quad dv = dx$$
$$du = \tfrac{2x-1}{x^2-x+2}dx \quad v = x$$

$$= x\ln(x^2 - x + 2) - \int \left(\frac{2x^2 - x}{x^2 - x + 2}\right)dx$$

$$= x\ln(x^2 - x + 2) - \int \left(\frac{2x^2 - x - x + x + 4 - 4}{x^2 - x + 2}\right)$$

$$= x\ln(x^2 - x + 2) - \int \left(\frac{2x^2 - 2x + 4}{x^2 - x + 2} + \frac{x - 4}{x^2 - x + 2}\right)dx$$

$$= x\ln(x^2 - x + 2) - \int \left(\frac{2(x^2 - x + 2)}{x^2 - x + 2} + \frac{x - \frac{1}{2}}{x^2 - x + 2} - \frac{\frac{7}{2}}{x^2 - x + 2}\right)dx$$

$$= x\ln(x^2 - x + 2) - \left(\int (2)dx + \int \left(\frac{x - \frac{1}{2}}{x^2 - x + 2}\right)dx - \frac{7}{2}\int \left(\frac{1}{(x - \frac{1}{2})^2 + \frac{7}{4}}\right)dx\right)$$

$$= x\ln(x^2 - x + 2) - 2x - \frac{1}{2}\ln(x^2 - x + 2) + \sqrt{7}\arctan(\frac{2x - 1}{\sqrt{7}}) + C$$

$$= (x - \frac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\arctan(\frac{2x - 1}{\sqrt{7}}) + C$$

60.

$$\int \left(\frac{1}{1 - \cos(x)}\right)dx = \int \left(\frac{1}{1 - \cos(x)} \cdot \frac{1 + \cos(x)}{1 + \cos(x)}\right)dx$$

$$= \int \left(\frac{1 + \cos(x)}{1 - \cos^2(x)}\right)dx = \int \left(\frac{1 + \cos(x)}{\sin^2(x)}\right)dx = \int \left(\frac{1}{\sin^2(x)} + \frac{\cos(x)}{\sin^2(x)}\right)$$

$$= \int (\csc^2(x) + \csc(x)\cot(x))dx = -\cot(x) - \csc(x) + C$$

61.

$$\int \left(\frac{1}{3\sin(x) - 4\cos(x)}\right)dx$$

Let $t = \tan(\frac{x}{2})$
$\frac{2}{1+t^2}dt = dx$

$$= \int \left(\frac{1}{3\frac{2t}{1+t^2} - 4\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \right) dt = \int \left(\frac{2}{6t - 4 + 4t^2} \right) dt = \int \left(\frac{1}{2t^2 + 3t - 2} \right)$$

$$= \int \left(\frac{1}{2(t^2 + \frac{3}{2}t) - 2} \right) dt = \int \left(\frac{1}{2(t^2 + \frac{3}{2}t + \frac{9}{16} - \frac{9}{16}) - 2} \right) dt$$

$$= \int \left(\frac{1}{2(t^2 + \frac{3}{2}t + \frac{9}{16}) - 2 - \frac{9}{8}} \right) dt = \int \left(\frac{1}{2(t + \frac{3}{4})^2 - \frac{25}{8}} \right) dt$$

$$= -\frac{2}{5} \operatorname{arctanh}(\frac{4t+3}{5}) + C = -\frac{2}{5} \operatorname{arctanh} \left(\frac{4\tan(\frac{x}{2}) + 3}{5} \right) + C$$

62.

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{1 + \sin(x) - \cos(x)} \right) dx$$

Let $t = \tan(\frac{x}{2})$
$\frac{2}{1+t^2}dt = dx$

$$\overset{1}{=} \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{1}{1 + \frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \right) dt = \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{2}{1 + t^2 + 2t - 1 + t^2} \right) dt$$

1: Note the bound change. It comes from plugging the original bounds of the integral into the substitution formula. You may opt to not do this (though I recommend against it as it leads to more computation later on), but if you choose to not do so, you **MUST** denote somehow that you recognize that the bounds do indeed change. How might you do this? You may write something like $\int_{x=\frac{\pi}{3}}^{x=\frac{\pi}{2}} \left(\frac{1}{1+t^2+2t-1+t^2} \right) dt$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{2}{2t^2 + 2t} \right) dt = \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{1}{t(t+1)} \right) dt$$

<u>Aside:</u>

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$$

$$1 = A(t+1) + Bt$$

Let $t = -1$

$$1 = B(-1) \to B - 1$$

Let $t = 0$

$$1 = A(0+1) \to A = 1$$

So, we have that

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{1 + \sin(x) - \cos(x)} \right) dx = \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{1}{t(t+1)} \right) dt = \int_{\frac{1}{\sqrt{3}}}^{1} \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

$$= \ln|t| - \ln|t+1| \Big|_{\frac{1}{\sqrt{3}}}^{1} = \ln 1 - \ln 2 - \left(\ln\left(\frac{1}{\sqrt{3}}\right) - \ln\left(\frac{1}{\sqrt{3}} + 1\right) \right)$$

$$= -\ln 2 + \frac{1}{2}\ln 3 + \ln(1+\sqrt{3}) - \frac{1}{2}\ln 3 = \ln\left(\frac{1+\sqrt{3}}{2}\right)$$