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Math 31 Summer 2019

Exam 1 Solutions

June 18, 2019

1. Evaluate the integral

$$\int_0^1 \left(\frac{x+1}{\sqrt{1-x^2}}\right)dx$$

$$\begin{aligned} \text{Let } x &= \sin(t) \\ dx &= \cos(t)dt \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sin(t)+1}{\sqrt{1-\sin^2(t)}} \cdot \cos(t)\right)dt = \int_0^{\frac{\pi}{2}} \left(\frac{\cos(t)(\sin(t)+1)}{\cos(t)}\right)dt$$

$$= \int_0^{\frac{\pi}{2}} (\sin(t)+1)dt = t - \cos(t)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} + 1$$

2. Evaluate the limit.

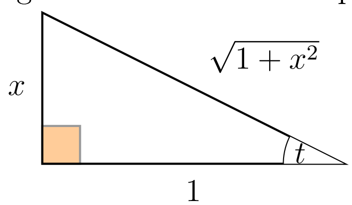
$$\lim_{x\to\infty} \sqrt{x}^{\frac{1}{\ln(x)}}$$

$$= \lim_{x\to\infty} e^{\ln(\sqrt{x}^{\frac{1}{\ln(x)}})} = \lim_{x\to\infty} e^{\ln(x^{\frac{1}{2\ln(x)}})} = e^{\lim_{x\to\infty} \frac{\ln(x)}{2\ln(x)}} = e^{\lim_{x\to\infty} \frac{1}{2}} = e^{\frac{1}{2}}$$

3. Rewrite $\sin(2\arctan(x))$ as an expression solely in x.

First recall the fact that $\sin(2x) = 2\sin(x)\cos(x)$

To proceed, we will assume that $\arctan(x) = t$ so we may construct a triangle. Notice that this implies $\tan(t) = x = \frac{x}{1}$.



So, based on our triangle we have that $\sin(t) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(t) = \frac{1}{\sqrt{1+x^2}}$.
But, recall we said $\arctan(x) = t$. Thus, we have:

$$\sin(2\arctan(x)) = \sin(2t) = 2\sin(t)\cos(t) = 2\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

4. Determine the volume generated by the region bounded by the given curves when revolved about the given axis:

$$\begin{cases} y = \frac{1}{x^2} \\ x = 1 \\ x = 2 \\ y = 0 \end{cases}$$

about the y-axis

$$V = 2\pi \int_1^2 \left(x \cdot \frac{1}{x^2} \right) dx = 2\pi \ln|x| \Big|_1^2 = 2\pi \ln 2$$

5a. Find y' for

$$y = e^{\sin^2(\ln x)}$$

$$y' = e^{\sin^2(\ln x)} \cdot 2\sin(\ln x) \cdot \cos(\ln x) \cdot \frac{1}{x}$$

$$y' \stackrel{1}{=} \frac{e^{\sin^2(\ln x)} \sin(2\ln x)}{x}$$

1: Note we used the fact $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. If you let $\theta = \ln x$ the reduction applied is apparent.

5b. Find y' for

$$y = x^{\ln^2 x}$$

$$\ln y = \ln(x^{\ln^2 x}) = \ln^2 x \cdot \ln x = \ln^3 x$$

$$\frac{1}{y}y' = \frac{3\ln^2 x}{x}$$

$$y' = \frac{3y\ln^2 x}{x} = \frac{3x^{\ln^2 x}\ln^2 x}{x} = 3x^{\ln^2(x)-1}\ln^2 x$$

6a. Evaluate the following integral

$$\int_0^{\frac{\pi}{2}} \left(\frac{\cos(x)}{\sqrt{\sin^2(x)+1}}\right)dx$$

Let $u = \sin(x)$
$du = \cos(x)dx$

$$= \int_0^1 \left(\frac{1}{\sqrt{u^2+1}}\right)du$$

Let $u = \tan(t)$
$du = \sec^2(t)dt$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sqrt{\tan^2(t)+1}} \cdot \sec^2(t)\right)dt = \int_0^{\frac{\pi}{4}} \left(\frac{\sec^2(t)}{\sqrt{\sec^2(t)}}\right)dt = \int_0^{\frac{\pi}{4}} (\sec(t))dt$$

$$= \ln|\sec(t) + \tan(t)|\Big|_0^{\frac{\pi}{4}} = \ln(\sqrt{2}+1) - \ln(1+0) = \ln(1+\sqrt{2})$$

6b. Evaluate the following integral

$$\int_0^1 \left(\frac{x}{1+x^4}\right)dx$$

Let $u = x^2$
$du = 2xdx$
$\frac{1}{2}du = xdx$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{1+u^2}\right) du = \frac{1}{2}\arctan(u)\Big|_0^1 = \frac{\pi}{8}$$

6c Evaluate the following integral

$$\int \left(\frac{1}{x\sqrt{1-\ln^2 x}}\right) dx$$

Let $u = \ln x$
$du = \frac{1}{x}dx$

$$= \int \left(\frac{1}{\sqrt{1-u^2}}\right) du$$

Let $u = \sin(t)$
$du = \cos(t)dt$

$$= \int \left(\frac{1}{\sqrt{1-\sin^2(t)}} \cdot \cos(t)\right) dt = \int \left(\frac{\cos(t)}{\sqrt{\cos^2(t)}}\right) dt = \int (1)dt = t + C$$
$$= \arcsin(u) + C = \arcsin(\ln x) + C$$

7. If $f(x) = 3x^3 + 5x + 11$ and $g = f^{-1}$, then find $g'(3)$.

Notice that since g and f are invertible, we have that $g(f(x)) = x$. But this means that if we different, we obtain the following:

$$g'(f(x)) \cdot f'(x) = 1 \to g'(f(x)) = \frac{1}{f'(x)}$$

$$g'(f(x)) = \frac{1}{9x^2+5}$$

Thus, to find $g'(3)$ it should be apparent that we need to solve for $3 = f(x)$ and then we plug into the formula we just found and we are done.

$$3 = f(x) = 3x^3 + 5x + 11$$

$$0 = 3x^3 + 5x + 8$$
$$0 = (x+1)(3x^2 - 3x + 8)$$

So, $x = -1$ or $x = \frac{3 \pm i\sqrt{87}}{6}$. We will ignore the complex solutions. Thus, we find that $x = -1$ is our desired value.

Thus, $g'(3) = g'(f(-1)) = \frac{1}{9(-1)^2+5} = \frac{1}{14}$.

8. Find y' if $y = \pi^x + x^\pi + e^\pi$.

Recall the useful formula $\frac{d}{dx}[a^x] = a^x \ln a, a \neq 0$. Notice how a is constant in this case.

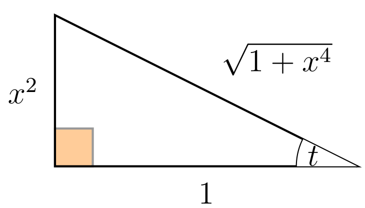
Next, recall $\frac{d}{dx}[x^n] = nx^{n-1}, n \neq 0$. Notice how in this case, n is a constant.

Lastly, also notice that e^π is nothing more than a number raised to another number and is thus a constant, so its derivative is 0.

Thus, we have that $y' = \pi^x \ln(\pi) + \pi x^{\pi-1}$

9. Rewrite $\sin(\arctan(x^2))$ as an expression in x (i.e. rewrite this expression without using any trigonometric or inverse trigonometric functions).

To proceed, we will assume that $\arctan(x^2) = t$ so we may construct a triangle. Notice that this implies $\tan(t) = x^2 = \frac{x^2}{1}$.



Now that this triangle is constructed, we find that $\sin(t) = \frac{x^2}{\sqrt{1+x^4}}$. Since we assumed $t = \arctan(x^2)$ we have $\sin(\arctan(x^2)) = \sin(t) = \frac{x^2}{\sqrt{1+x^4}}$