Consider the following:

1. Consider the two polar equations: $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$

   i) Sketch the graph of these two polar equations in the same coordinate axis system.

   ii) Find the values of θ that generate the two points of intersection between these two curves.

   iii) If we concentrate merely on the images within the first quadrant, determine the area that lies within the graph of $r^2 = 2\sin\theta$ and outside the graph of $r = 2\cos\theta$. Recall that:

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left((\text{outer radius})^2 - (\text{inner radius})^2\right)d\theta$$

   iv) In the *xy*-plane, the lower edge of the bounded region whose area you found in part iii is an arc on one of the graphs. Determine the length of this arc. The means by which we compute the length of an arc described by a polar equation is $L = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

   v) At the one point of intersection between these two curves that is in the first quadrant, are these curves orthogonal (i.e., are their tangent lines perpendicular)? The means by which we compute slopes of tangent lines for polar equations is $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$

2. i) Clearly sketch the graph of $r = \frac{1}{1 - 2\cos\theta}$.

   ii) Rewrite this polar equation as a rectangular equation by recalling (from trigonometry) that $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

3. The graph of the equation $r = \theta$ is known as the *Spiral of Archimedes*.

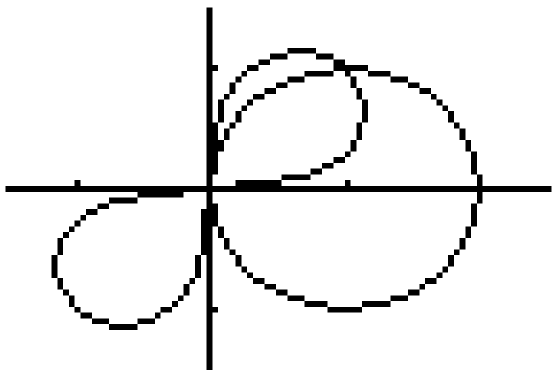
i) Create a rough sketch or the polar graph of this equation.

ii) Let A_n represent the area bounded by the nth term of the spiral. Show that $A_1 = \frac{4}{3}\pi^3$ and $A_2 = \frac{28}{3}\pi^3$. Recall that $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

Consider the following:

1. Consider the two polar equations: $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$

i) Sketch the graph of these two polar equations in the same coordinate axis system.



ii) Find the values of θ that generate the two points of intersection between these two curves.

We will set the *r* values equal to each other and solve for θ:

$$\sqrt{2\sin 2\theta} = 2\cos\theta$$
$$2\sin 2\theta = 4\cos^2\theta$$
$$4\sin\theta cos\theta = 4\cos^2\theta$$
$$4\sin\theta cos\theta - 4\cos^2\theta = 0$$
$$\cos\theta(\sin\theta - \cos\theta) = 0$$
$$\cos\theta = 0 \quad \text{or} \quad \sin\theta - \cos\theta = 0$$
$$\cos\theta = 0 \quad \text{or} \quad \sin\theta = \cos\theta$$
$$\cos\theta = 0 \quad \text{or} \quad \tan\theta = 1$$
$$\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{4}$$

iii) If we concentrate merely on the images within the first quadrant, determine the area that lies within the graph of $r^2 = 2\sin\theta$ and outside the graph of $r = 2\cos\theta$. Recall that:

$$A = \frac{1}{2}\int_{\alpha}^{\beta}\left(\left(\text{outer radius}\right)^2 - \left(\text{inner radius}\right)^2\right)d\theta$$

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(2\sin 2\theta - \left(2\cos\theta\right)^2\right)d\theta$$

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(2\sin 2\theta - 4\cos^2\theta\right)d\theta$$

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(2\sin 2\theta - 2\left(1+\cos 2\theta\right)\right)d\theta$$

$$A = \frac{1}{2}\int_{\pi/4}^{\pi/2}\left(2\sin 2\theta - 2 - 2\cos 2\theta\right)d\theta$$

$$A = \frac{1}{2}\left(-\cos 2\theta - 2\theta - \sin 2\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$A = -\frac{1}{2}\left(\cos 2\theta + 2\theta + \sin 2\theta\right)\Big|_{\pi/4}^{\pi/2}$$

$$A = -\frac{1}{2}\left(-1 + \pi + 0 - \left(0 + \frac{\pi}{2} + 1\right)\right)$$

$$A = -\frac{1}{2}\left(-2 + \frac{\pi}{2}\right)$$

$$A = 1 - \frac{\pi}{4}$$

iv) In the *xy*-plane, the lower edge of the bounded region whose area you found in part iii is an arc on one of the graphs. Determine the length of this arc. The means by which we compute the length of an arc described by a polar equation is $L = \int_{\alpha}^{\beta}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{4\cos^2\theta + 4\sin^2\theta}\, d\theta$$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{4\left(\cos^2\theta + \sin^2\theta\right)}\, d\theta$$

$$L = \int_{\pi/4}^{\pi/2} \sqrt{4}\sqrt{\cos^2\theta + \sin^2\theta}\, d\theta$$

$$L = \int_{\pi/4}^{\pi/2} 2\, d\theta$$

$$L = 2\theta \Big|_{\pi/4}^{\pi/2}$$

$$L = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$L = \frac{\pi}{2}$$

v) At the one point of intersection between these two curves that is in the first quadrant, are these curves orthogonal (i.e., are their tangent lines perpendicular)? The means by which we compute slopes of tangent lines for polar equations is $\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$. So, we have the following:

For the first curve,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\frac{dy}{dx} = \frac{4\cos 2\theta \sin\theta + 2\sin 2\theta \cos\theta}{4\cos 2\theta \cos\theta - 2\sin 2\theta \sin\theta}$$

And at $\theta = \frac{\pi}{4}$, the slope of the tangent line to this curve is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{4\cdot 0\cdot\frac{\sqrt{2}}{2}+2\cdot 1\cdot\frac{\sqrt{2}}{2}}{4\cdot 0\cdot\frac{\sqrt{2}}{2}-2\cdot 1\cdot\frac{\sqrt{2}}{2}} = -1$$

For the second curve,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

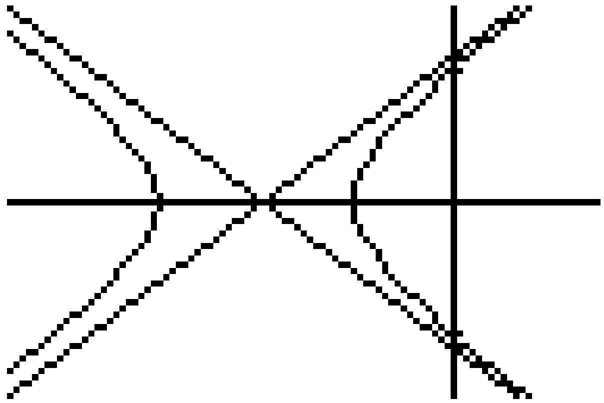
$$\frac{dy}{dx} = \frac{-2\sin\theta\sin\theta + 2\cos\theta\cos\theta}{-2\sin\theta\cos\theta - 2\cos\theta\sin\theta}$$

And at $\theta = \frac{\pi}{4}$, the slope of the tangent line to this curve is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{-2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}+2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}}{-2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}-2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}} = \frac{-2\cdot\frac{1}{2}+2\cdot\frac{1}{2}}{-2\cdot\frac{1}{2}-2\cdot\frac{1}{2}} = 0$$

Consequently, these curves are indeed not orthogonal at $\theta = \frac{\pi}{4}$, for the slopes of the tangent lines are not negative reciprocals of each other.

2. i)



ii) Rewrite this polar equation as a rectangular equation by recalling (from trigonometry) that $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2 + y^2}$

$$
\begin{aligned}
r &= \frac{1}{1-2\cos\theta} \\
r\left(1-2\cos\theta\right) &= 1 \\
r - 2r\cos\theta &= 1 \\
\sqrt{x^2+y^2} - 2x &= 1 \\
\sqrt{x^2+y^2} &= 2x+1 \\
x^2 + y^2 &= \left(2x+1\right)^2 \\
x^2 + y^2 &= 4x^2 + 4x + 1 \\
y^2 &= 3x^2 + 4x + 1 \\
y^2 &= 3\left(x^2 + \frac{4}{3}x\right) + 1
\end{aligned}
$$

$$y^2 = 3\left(x^2 + \frac{4}{3}x + \frac{4}{9}\right) + 1 - \frac{4}{3}$$

$$y^2 = 3\left(x + \frac{2}{3}\right)^2 - \frac{1}{3}$$

$$3\left(x + \frac{2}{3}\right)^2 - y^2 = \frac{1}{3}$$

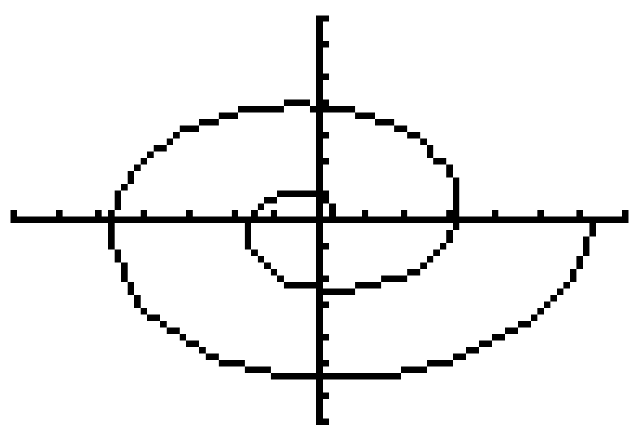
$9\left(x + \frac{2}{3}\right)^2 - 3y^2 = 1$, or more traditionally,

$$\frac{\left(x + \frac{2}{3}\right)^2}{1/9} - \frac{y^2}{1/3} = 1$$

Which clearly describes in rectangular form a hyperbola whose center is located at $\left(-\frac{2}{3}, 0\right)$ and branches out in the horizontal direction.

3. The graph of the equation $r = \theta$ is known as the *Spiral of Archimedes*.

i) Create a rough sketch or the polar graph of this equation.



ii) Let A_n represent the area bounded by the nth term of the spiral. Show that $A_1 = \frac{4}{3}\pi^3$ and $A_2 = \frac{28}{3}\pi^3$. Recall that $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

First, we have that $A_1 = \frac{1}{2}\int_0^{2\pi} r^2 d\theta$

$$A_1 = \frac{1}{2}\int_0^{2\pi} \theta^2 d\theta$$

$$A_1 = \frac{1}{2}\cdot\frac{1}{3}\theta^3 \Big|_0^{2\pi}$$

$$A_1 = \frac{1}{2}\cdot\frac{1}{3}\cdot(2\pi)^3$$

$$A_1 = \frac{4}{3}\pi^3 \qquad (*)$$

Secondly, we anticipate that $A_2 = \frac{1}{2}\int_0^{4\pi} r^2 d\theta$. Note, however, that $A_2 \neq \frac{1}{2}\int_0^{4\pi} \theta^2 d\theta$, for this counts the area of the first turn twice.

Consequently, we have that:

$$A_2 = \frac{1}{2}\int_0^{4\pi} \theta^2 d\theta - (*)$$

$$A_2 = \frac{1}{2}\int_0^{4\pi} \theta^2 d\theta - \frac{4}{3}\pi^3$$

$$A_2 = \frac{1}{2}\cdot\frac{1}{3}\theta^3 \Big|_0^{4\pi} - \frac{4}{3}\pi^3$$

$$A_2 = \frac{1}{2}\cdot\frac{1}{3}\cdot(4\pi)^3 - \frac{4}{3}\pi^3$$

$$A_2 = \frac{32}{3}\pi^3 - \frac{4}{3}\pi^3$$

$$A_2 = \frac{28}{3}\pi^3$$