1. Use the ε, δ definition of the limit to prove $\lim_{x \to 5}(2x - 3) = 7$

- Part 1: Find a δ that works:

  Given $\varepsilon > 0$, we need to find a δ such that :

$$|2x - 3 - 7| < \varepsilon \quad \text{whenever} \quad 0 < |x - 5| < \delta$$
$$|2x - 10| < \varepsilon \quad \text{whenever} \quad "$$
$$|2(x - 5)| < \varepsilon \quad \text{whenever} \quad "$$
$$2|x - 5| < \varepsilon \quad \text{whenever} \quad "$$
$$|x - 5| < \varepsilon/2 \quad \text{whenever} \quad "$$

  So, choose $\delta = \varepsilon/2$

- Part 2: Show that this δ works:

  Given $\varepsilon > 0$, choose $\delta = \varepsilon/2$

  If $0 < |x - 5| < \delta$, then
$$\begin{aligned} |2x - 3 - 7| &= |2x - 10| \\ &= |2(x - 5)| \\ &= 2|x - 5| \\ &< 2 \cdot \delta \\ &= 2 \cdot \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

/ $\therefore \lim_{x \to 5}(2x - 3) = 7$

2. Prove that $\lim_{x\to 3}(x^2+2x-5)=10$:

Part I: Find a δ that works:

Given an $\varepsilon>0$, we need a $\delta>0$ such that

$$\left|(x^2+2x-5)-10\right|<\varepsilon \text{ whenever } 0<|x-3|<\delta$$

But $\left|(x^2+2x-5)-10\right|<\varepsilon$ whenever $0<|x-3|<\delta$

is equivalent to:

$$|x^2+2x-15|<\varepsilon \qquad \text{whenever } 0<|x-3|<\delta$$

$$|(x+5)(x-3)|<\varepsilon \qquad \text{whenever } 0<|x-3|<\delta$$

$$|x+5|\cdot|x-3|<\varepsilon \qquad \text{whenever } 0<|x-3|<\delta$$

If we can find some number C such that $|x+5|<C$, then we will have that

$|x-3|<\varepsilon/C$, and we will choose $\delta=\varepsilon/C$.

Aside: Find C:

A reasonable choice for δ is 1. If we choose δ=1, then we have that:

$$0<|x-3|<1$$

$$-1<x-3<1$$

$$7<x+5<9$$

$$|x+5|<9$$

So, we now have a value for C to be 9. Thus, an alternative to 1 as a choice for δ is $\varepsilon/9$.

Now that we have two values for δ, to ensure that we obtain the desired tolerance value ε, we will choose $\delta=\min\{1,\varepsilon/9\}$.

Part II: Show that our choice for δ works:

Given ε>0, choose $\delta=\min\{1,\varepsilon/9\}$.

If $0<|x-3|<\delta$, then $\left|(x^2+2x-5)-10\right|=|x^2+2x-15|$

$$=|(x+5)(x-3)|$$

$$=|x+5|\cdot|x-3|$$

Note: If $\delta=1$, then $0<|x-3|<1$

which implies $-1<x-3<1$

i.e., $7<x+5<9$

or $|x+5|<9$

But, if $\delta=\varepsilon/9$, then we have that $0<|x-3|<\varepsilon/9$

Now we have that $\left|(x^2+2x-5)-10\right|=|x+5|\cdot|x-3|$

$$<9\cdot\varepsilon/9$$

$$=\varepsilon$$

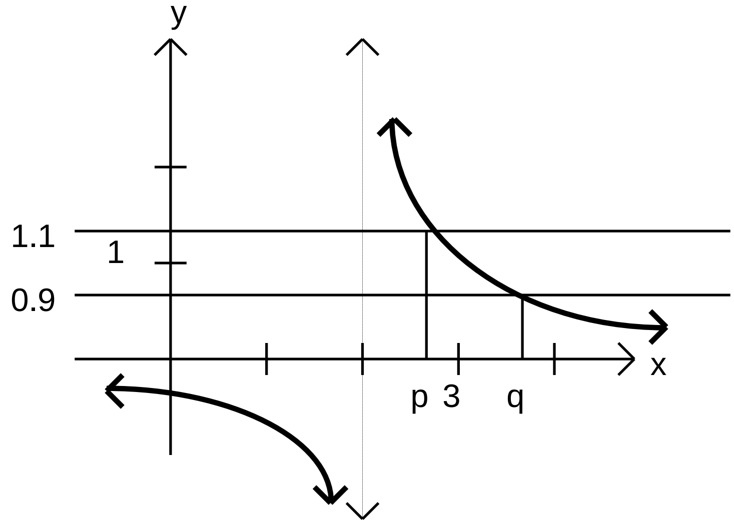
We have shown that If $0<|x-3|<\delta$, then $\left|(x^2+2x-5)-10\right|<\varepsilon$

/∴ $\lim_{x\to 3}(x^2+2x-5)=10$

3. In answering the following problem, round all values to the nearest 0.01.

Given $f(x)=\dfrac{1}{x-2}$, $\lim_{x\to 3} f(x)=1$, and $\varepsilon=0.1$, find the largest value of δ such that

If $0<|x-3|<\delta$, then $|f(x)-1|<\varepsilon$. Draw a picture to support your claim.



- Note: To find the values of *p* and *q* in the above picture, solve the equations:

$$\frac{1}{p-2}=1.1 \qquad \frac{1}{q-2}=0.9$$

$$\frac{1}{1.1}=p-2 \qquad \frac{1}{0.9}=q-2$$

$$p=2+\frac{1}{1.1} \qquad q=2+\frac{1}{0.9}$$

$$p\approx 2.91 \qquad q\approx 3.11$$

- Choose δ as follows:

$$\begin{aligned}\delta &= \min\{|3-p|,|3-q|\}\\ &= \min\{0.09, 0.11\}\\ &= 0.09\end{aligned}$$

Consider that the above suggests that if a value of *x* is chosen within 0.09 units from 3, *f*(*x*) is guaranteed to be within 0.1 units of 1.

4. Use the ε,δ definition of the limit to prove $\lim_{x\to 5}(3x-4)=11$

- Part 1: Find a δ that works:

  Given $\varepsilon > 0$, we need to find a δ such that :

  $$|3x-4-11| < \varepsilon \quad \text{whenever} \quad 0 < |x-5| < \delta$$
  $$|3x-15| < \varepsilon \quad \text{whenever} \quad \text{"}$$
  $$|3(x-5)| < \varepsilon \quad \text{whenever} \quad \text{"}$$
  $$3|x-5| < \varepsilon \quad \text{whenever} \quad \text{"}$$
  $$|x-5| < \varepsilon/3 \quad \text{whenever} \quad \text{"}$$

  So, choose $\delta = \varepsilon/3$

- Part 2: Show that this δ works:

  Given $\varepsilon > 0$, choose $\delta = \varepsilon/3$

  If $0 < |x-5| < \delta$, then
  $$\begin{aligned} |3x-4-11| &= |3x-15| \\ &= |3(x-5)| \\ &= 3|x-5| \\ &< 3\cdot\delta \\ &= 3\cdot\varepsilon/3 \\ &= \varepsilon \end{aligned}$$

/ $\therefore \lim_{x\to 5}(3x-4)=11$

5. Prove that $\lim_{x\to 2}(x^2-3x+3)=1$:

Part I: Find a δ that works:
Given an ε>0, we need a δ>0 such that

$$\left|\left(x^2-3x+3\right)-1\right|<\varepsilon \text{ whenever } 0<|x-2|<\delta$$

But $\left|\left(x^2-3x+3\right)-1\right|<\varepsilon$ is equivalent to:

$$\left|x^2-3x+2\right|<\varepsilon$$
$$\left|(x-1)(x-2)\right|<\varepsilon$$
$$|x-1|\cdot|x-2|<\varepsilon$$

If we can find some number C such that $|x-1|<C$, then we will have that $|x-2|<\varepsilon/C$, and we will choose $\delta=\varepsilon/C$.

Aside: Find C:

A reasonable choice for δ is 1. If we choose δ=1, then we have that:

$$0<|x-2|<1$$
$$-1<x-2<1$$
$$0<x-1<2$$
$$|x-1|<2$$

So, we now have a value for C to be 2. Thus, an alternative to 1 as a choice for δ is $\varepsilon/2$.

Now that we have two values for δ, to ensure that we obtain the desired tolerance value ε, we will choose $\delta=\min\{1,\varepsilon/2\}$.

<u>Part II</u>: Show that our choice for δ works:

Given ε>0, choose $\delta=\min\{1,\varepsilon/2\}$.

If $0<|x-2|<\delta$, then $\left|\left(x^2-3x+3\right)-1\right|=\left|x^2-3x+2\right|$

$$=\left|(x-1)(x-2)\right|$$
$$=|x-1|\cdot|x-2|$$

Note: If δ=1, then $0<|x-2|<1$

Which implies $-1<x-2<1$

i.e., $0<x-1<2$

or $|x-1|<2$

But, if $\delta=\varepsilon/2$, then we have that $0<|x-2|<\varepsilon/2$

Now we have that $\left|\left(x^2-3x+3\right)-1\right|=|x-1|\cdot|x-2|$

$$<2\cdot\varepsilon/2$$
$$=\varepsilon$$

We have shown that If $0<|x-2|<\delta$, then $\left|\left(x^2-3x+3\right)-1\right|<\varepsilon$

/∴ $\lim_{x\to 3}\left(x^2-3x+3\right)=1$

6. Prove that $\lim_{x\to 2}\left(2x^2-x-2\right)=4$:

<u>Part I</u>: Find a δ that works:

Given an $\varepsilon>0$, we need a $\delta>0$ such that

$\left|\left(2x^2 - x - 2\right) - 4\right| < \varepsilon$ whenever $0 < |x-2| < \delta$

But $\left|\left(2x^2 - x - 2\right) - 4\right| < \varepsilon$ whenever $0 < |x-2| < \delta$

is equivalent to:

$\left|2x^2 - x - 6\right| < \varepsilon$ whenever $0 < |x-2| < \delta$

$\left|(2x+3)(x-2)\right| < \varepsilon$ whenever $0 < |x-2| < \delta$

$|2x+3||x-2| < \varepsilon$ whenever $0 < |x-2| < \delta$

If we can find some number C such that $|2x+3| < C$, then we will have that

$|x-2| < \varepsilon/C$, and we will choose $\delta = \varepsilon/C$.

Aside: Find C:

A reasonable choice for δ is 1. If we choose $\delta = 1$, then we have that:

$$0 < |x-2| < 1$$
$$-1 < x - 2 < 1$$
$$-2 < 2x - 4 < 2$$
$$5 < 2x + 3 < 9$$
$$|2x+3| < 9$$

So, we now have a value for C to be 9. Thus, an alternative to 1 as a choice for δ is $\varepsilon/9$.

Now that we have two values for δ, to ensure that we obtain the desired tolerance value ε, we will choose $\delta = \min\{1, \varepsilon/9\}$.

<u>Part II</u>: Show that our choice for δ works:

Given $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/9\}$.

If $0 < |x-2| < \delta$, then $\left|\left(2x^2 - x - 2\right) - 4\right| = \left|2x^2 - x - 6\right|$

$$= \left|(2x+3)(x-2)\right|$$
$$= |2x+3| \cdot |x-2|$$

Note: If $\delta = 1$, then $0 < |x-2| < 1$

which implies $-1 < x - 2 < 1$

so $-2 < 2x - 4 < 2$

i.e., $5 < 2x + 3 < 9$

or $|2x+3| < 9$

But, if $\delta = \varepsilon/9$, then we have that $0 < |x-2| < \varepsilon/9$

Now we have that $\left|\left(2x^2 - x - 2\right) - 4\right| = |2x+3| \cdot |x-2|$

$$< 9 \cdot \varepsilon/9$$
$$= \varepsilon$$

We have shown that If $0 < |x-2| < \delta$, then $\left|\left(2x^2 - x - 2\right) - 4\right| < \varepsilon$

$/ \therefore \lim_{x \to 2}\left(2x^2 - x - 2\right) = 4$