Provide a clear and organized presentation. Show all of your work, completely simplify your answers, and give exact values only.

1.   (10 pts) Consider the following:

   i)   Given $a = p^c q^d r^e$ and $b = p^d q^e r^c$ where all letters are natural numbers and $d < c < e$. Determine both the gcd and the lcm of a and b.

   ii)   Express the lcm of two natural numbers m and n in terms of their gcd.

   iii)   Under what circumstances is $\gcd(m, n) = \text{lcm}(m, n) = p^2$ where m and n are two natural numbers and p is prime.

2.   (5 pts) Solve the following system using Gauss-Jordan elimination:

$$2x_1 - x_2 + 3x_3 + x_4 - 2x_5 = 1$$
$$3x_1 + x_2 - x_3 + 2x_4 + 2x_5 = 1$$
$$4x_1 - 7x_2 + 17x_3 + x_4 - 14x_5 = 1$$

3.   (10 pts) Consider the set $S = \mathbb{N} \setminus \{1\}$ where aRb means that a and b are not relatively prime (i.e., a and b are relatively prime if a and b share no common prime factors)

   i)   Is this relation reflexive? Symmetric? Transitive?

   ii)   Does this relation partition S?

4.   (10 pts) Use a generating function to rewrite the following recurrence relation as an   explicit formula for the nth term of the sequence:

$$a_n = a_{n-2} - a_{n-1} \text{ where } a_0 = 0 \text{ and } a_1 = 1$$

5.   (5 pts) Prove that the identity element of a group is unique.