Provide both a clear and organized presentation. Show all of your work, completely simplify your answers, and use exact values only. No technology, other than a scientific calculator, can be used to complete this exam.

1. (25 pts) Solve the differential equation: $y^3\left(x^3+8\right)\dfrac{dy}{dx}=\left(x^4-8\right)\left(y^2+2\right)^{3/2}$

This is indeed separable: $\displaystyle\int\frac{y^3}{\left(y^2+2\right)^{3/2}}dy=\int\frac{x^4-8}{x^3+8}dx$

We will commit to a trig sub on the left and long division (add zero) and pfd on the right:

Let $y=\sqrt{2}\tan\theta$

$dy=\sqrt{2}\sec\theta d\theta$

$$\sqrt{2}\int\frac{\sin^3\theta}{\cos^2\theta}d\theta=\int\frac{x^4+8x-8x-8}{x^3+8}dx$$

$$\sqrt{2}\int\frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta=\int\left(x+\frac{\frac{2}{3}}{x+2}+\frac{-\frac{2}{3}x-\frac{16}{3}}{x^2-2x+4}\right)dx$$

$$\sqrt{2}\int\left(1-\frac{1}{u^2}\right)du=\frac{1}{2}x^2+\frac{2}{3}\ln\left|x+2\right|+\int\left(\frac{-\frac{2}{3}x+\frac{2}{3}}{x^2-2x+4}+\frac{-\frac{18}{3}}{\left(x-1\right)^2+3}\right)dx$$

$$\frac{y^2+4}{\sqrt{y^2+2}}=\frac{1}{2}x^2+\frac{1}{3}\ln\frac{x^2+4x+4}{x^2-2x+4}-2\sqrt{3}\tan^{-1}\frac{x-1}{\sqrt{3}}+C$$

2. (10 pts) Use isoclines to draw a slope field for the differential equation $y' = \frac{y-2}{x}$. Draw and clearly label in your slope field any equilibrium solutions. In addition, draw a couple of additional solutions. What does the existence-uniqueness theorem say about a solution passing through the origin? What does it say about a solution passing through any point other than the origin?

3. (10 pts) Determine the family of orthogonal trajectories to the family of curves described by the following: $y = k\cos x$

$$\frac{dy}{dx} = -k\sin x$$

$$\frac{dy}{dx} = \frac{1}{k\sin x} \text{ where } k = \frac{y}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x}{y\sin x}$$

$$\int y\,dy = \int \cot x\,dx$$

$$y^2 = \ln\sin^2 x + C_1$$

4. (15 pts) Solve the differential equation $1+\left(y'\right)^2 + y \cdot y'' = 0$

Let $u = y'$
and $u \cdot u' = y''$ giving us:

$1+u^2 + y \cdot u \cdot u' = 0$ which is clearly a Bernouilli D.E.

Letting $v = u^2$ yields $\frac{1}{2}\frac{dv}{dy} + \frac{1}{y} \cdot v = -\frac{1}{y}$

Consequently, $u = \pm\sqrt{C_1 y^{-2} - 1}$

$$\int \frac{1}{\pm\sqrt{C_1 y^{-2} - 1}} dy = \int dx$$

Finally, we have that $\left(x \pm C_2\right)^2 + y^2 = C_1$

5. (15 pts) Solve the following differential equation:

$$\ln\left(y+\sqrt{y^2+x^2}\right)-\ln x=\frac{x}{xy'-y} \quad \text{where } x>0, y>0$$

Note that $\ln\left(\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1}\right)=\frac{1}{y'-\frac{y}{x}}$ and let $u=\frac{y}{x}$

$\int \ln\left(u+\sqrt{u^2+1}\right)du=\int \frac{1}{x}dx$ and after a bit of integration by parts activity,

$\frac{y}{x}\ln\left(\frac{y}{x}+\sqrt{\frac{y^2}{x^2}+1}\right)-\sqrt{\frac{y^2}{x^2}+1}=\ln x+C$, or more simply,

$y\ln\left(y+\sqrt{y^2+x^2}\right)-(x+y)\ln x-x\sqrt{y^2+x^2}=C\cdot x$

6. (15 pts) Solve the differential equation $\frac{dy}{dx} = xy + \frac{x^3}{y}$

Clearly a Bernouilli D.E., we will commit to $u = y^2$

$$\frac{du}{dx} - 2xu = 2x^3 \text{ with } I(x) = e^{-x^2}$$
$$ue^{-x^2} = \int 2x^3 e^{-x^2} dx$$
$$y^2 = Ce^{x^2} - x^2 - 1$$

7. (10 pts) A farmer's rice field is supplied by two sources of water. For a duration of time, runoff from other nearby irrigated fields supplies this rice field with 100 gallons/hr, but is polluted with 10 g/gallon of assorted pollutants. Another source is constant rain that supplies his field with 50 gallons/hr, but is clean. Water from his field is released at 100 gallons/hr. His field has the capacity to hold 100,000 gallons of water, but is only half full. If his field is initially pollutant-free, then determine a function that gives us the amount of pollutant in this field during the time that the field is filling (i.e., before it reaches its capacity).

$$\frac{dA}{dt} = 1000 - \frac{100A}{50{,}000 + 50t}$$

$$\frac{dA}{dt} + \frac{2A}{1000+t} = 1000 \text{ with an integrating factor } I(t) = (1000+2)^2$$

$$A(t) = \frac{1000}{3}(1000+t) + \frac{C}{(1000+t)^2} \text{ where } C = -\frac{1000^4}{3}$$

$$A(t) = \frac{1000}{3}(1000+t) - \frac{1}{3}\cdot\frac{1000^4}{(1000+t)^2}$$