

Show all of your work and completely simplify all of your answers.

1. (20 pts) Find the points on the surface $z^2 = xy + yz + xz - x$ where the tangent plane is parallel to the xy -plane. What are the equations of those tangent planes?

14.4 # 1-6, # 42 / 14.7 # 54, # 55-57 /

Chapter Review
31

2. (20 pts) Let $f(x,y) = x^2 \ln y$.

Chapter Review # 45 / 14.6 # 11-17, 21-26

- i) At the point $(2,e)$ in the xy -coordinate plane, what is the rate of change for f in the direction of $\langle 1,3e \rangle$?

- ii) At the point $(2,e)$, in which direction is the rate of change for f the greatest?

- iii) What is the maximum rate of change for f at the point $(2,e)$?

3. (15 pts) Determine the extrema for $f(x,y) = x^2 + 3y^2 + 2y$ subject to $x^2 + y^2 \leq 1$.

Chapter Review # 56 / 14.8 # 21 & 22

4. (15 pts) Determine the local extrema and saddle points for $f(x,y) = x^4 + y^3 + 32x - 6y$.

Chapter Review # 51-54 / 14.7 # 5-18 / 14.7 # 13

5. (15 pts) Find the points on the curve created by the intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$ that are closest to the origin.

Chapter Review # 63 / 14.8 # 17-20 / Example 4 pg 975

6. (15 pts) If $z = f(x,y)$ where $x = r \cos \theta$ and $y = r \sin \theta$ prove that

$$\left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2.$$

14.5 # 45 Exactly

$$\textcircled{1} \quad z^2 = xy + yz + xz - x \rightarrow \underbrace{xy + yz + xz - x - z^2}_F(x, y, z) = 0$$

$$F_x = y + z - 1 \quad F_y = x + z \quad F_z = y + x - 2z$$

Recall tangent plane is $F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$
 ? xy -plane is $z=0$ so we must have $F_x = F_y = 0$

$$0 = y + z - 1 \rightarrow y = 1 - z \quad 0 = x + z \rightarrow x = -z \quad \text{so..}$$

$$F(x, y, z) = F(-z, 1-z, z) = (-z)(1-z) + (1-z)z + (-z)(z) - (-z) - z^2 = 0$$

$$-z + z^2 + z - z^2 - z^2 + z - z^2 = 0$$

$$-2z^2 + z = 0$$

$$-z(2z - 1) = 0$$

$$z=0 \text{ or } z=\frac{1}{2}$$

$$\text{Case A} \quad z=0 \rightarrow x=0 \rightarrow y=1$$

$$\text{Case B} \quad z=\frac{1}{2} \rightarrow x=-\frac{1}{2} \rightarrow y=\frac{1}{2}$$

$$\boxed{\begin{pmatrix} 0, 1, 0 \\ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}}$$

$$\textcircled{A} \quad F_x(0, 1, 0) = 0 \quad F_y(0, 1, 0) = 0 \quad F_z(0, 1, 0) = 1$$

$$0(x-0) + 0(y-1) + 1(z-0) = 0 \rightarrow \boxed{z=0}$$

Neat the
surface is
tangent to
 xy -plane
at $(0, 1, 0)$

$$\textcircled{B} \quad F_x\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 0 \quad F_y\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = 0 \quad F_z\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = -1$$

$$0(x+\frac{1}{2}) + 0(x-\frac{1}{2}) - 1(z-\frac{1}{2}) = 0 \rightarrow \boxed{z = \frac{1}{2}}$$

$$(2) \quad f(x,y) = x^2 \ln y$$

$$f_x = 2x \ln y \quad f_y = x^2 \left(\frac{1}{y} \right) = \frac{x^2}{y}$$

$$f_x(2,e) = 2(2) \ln(e) = 4 \quad f_y(2,e) = \frac{(2)^2}{e} = \frac{4}{e}$$

i) $\nabla = \langle 1, 3e \rangle \quad |\nabla| = \sqrt{1+9e^2} \quad \text{so} \quad \vec{u} = \left\langle \frac{1}{\sqrt{1+9e^2}}, \frac{3e}{\sqrt{1+9e^2}} \right\rangle$

$$\begin{aligned} D_u f(2,3) &= \nabla f(2,3) \cdot \vec{u} = \langle 4, \frac{4}{e} \rangle \cdot \left\langle \frac{1}{\sqrt{1+9e^2}}, \frac{3e}{\sqrt{1+9e^2}} \right\rangle \\ &= \frac{4}{\sqrt{1+9e^2}} + \frac{12}{\sqrt{1+9e^2}} = \boxed{\frac{16}{\sqrt{1+9e^2}}} \end{aligned}$$

ii) maximum of $D_u f(2,3)$ occurs when $\vec{u} = c \nabla f$
so max will be in the direction of $\langle 4, \frac{4}{e} \rangle$

iii) maximum of $D_u f(2,3)$ is $|\nabla f(2,3)|$ which is

$$\sqrt{(4)^2 + \left(\frac{4}{e}\right)^2} = \sqrt{16 + \frac{16}{e^2}} = \boxed{\frac{4\sqrt{1+e^2}}{e}}$$

③ First let's consider $f(x, y) = x^2 + 3y^2 + 2y$
 given $\underbrace{x^2 + y^2 = 1}_{g(x, y)}$

Using LaGrange we have $\nabla f = \lambda \nabla g$

$$\nabla f = \langle 2x, 6y + 2 \rangle = \lambda \langle 2x, 2y \rangle = \lambda \nabla g$$

$$2x = \lambda 2x \rightarrow 2x(1-\lambda) = 0 \\ 6y + 2 = \lambda 2y \quad x=0 \text{ or } \lambda = 1 \\ x^2 + y^2 = 1$$

Case A) $x=0 \rightarrow (0)^2 + y^2 = 1 \rightarrow y = \pm 1$ ($\lambda = 4 \text{ or } \lambda = 2$)

Case B) $\lambda = 1 \rightarrow 6y + 2 = 2y \rightarrow y = -\frac{1}{2} \text{ so } x = \pm \frac{\sqrt{3}}{2}$

Critical values (so far...)

$$(0, 1) / (0, -1) / \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) / \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Now for the "interior" $f_x = 2x \quad f_y = 6y + 2$
 $0 = 2x \quad 0 = 6y + 2$
 Another critical value $(0, \frac{1}{3}) \quad x = 0 \quad y = -\frac{1}{3}$

$f_{xx} = 2 > 0 \quad f_{yy} = 6 \quad f_{xy} = 0$ so since $(2)(6) - (0)^2 > 0$
 this generates a local minimum

Now onto arithmetic conclusion

$$f(0, 1) = 5$$

$$f(0, -1) = 1$$

$$f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{3}{4} - 1 = \frac{1}{2}$$

$$f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) = \frac{3}{4} + \frac{3}{4} - 1 = \frac{1}{2}$$

$$f(0, -\frac{1}{3}) = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}$$

Local \ddagger , Absolute minimum of $-\frac{1}{3}$

$$(0, -\frac{1}{3}, -\frac{1}{3})$$

Absolute maximum of 5

$$(0, 1, 5)$$

No Local Maximums or saddle points.

$$(4) f(x, y) = x^4 + y^3 + 32x - 6y$$

$$f_x = 4x^3 + 32$$

$$\downarrow$$

$$0 = 4x^3 + 32$$

$$4x^3 = -32$$

$$x^3 = -8$$

$$x = -2$$

$$f_y = 3y^2 - 6$$

$$\downarrow$$

$$0 = 3y^2 - 6$$

$$6 = 3y^2$$

$$2 = y^2$$

$$y = \pm \sqrt{2}$$

Critical Values

$$(-2, \sqrt{2})$$

$$(-2, -\sqrt{2})$$

$$f_{xx} = 12x^2 > 0 \quad f_{yy} = 6y \quad f_{xy} = 0$$

$$@ (-2, \sqrt{2}) \text{ we have } D = (12(-2)^2)(6(\sqrt{2})) - (0)^2 > 0$$

Since $f_{xx} > 0$ & $D > 0$ Local Min

$$@ (-2, -\sqrt{2}) \text{ we have } D = (12(-2)^2)(6(-\sqrt{2})) - (0)^2 < 0$$

Saddle point

$$\text{Finally } f(-2, \sqrt{2}) = 16 + 2\sqrt{2} - 64 - 6\sqrt{2} = -48 - 4\sqrt{2}$$

$$f(-2, -\sqrt{2}) = 16 - 2\sqrt{2} - 64 + 6\sqrt{2} = -48 + 4\sqrt{2}$$

Local Min @ $(-2, \sqrt{2}, -48 - 4\sqrt{2})$

Saddle Point @ $(-2, -\sqrt{2}, -48 + 4\sqrt{2})$

No Local Max

⑤ Without La Grange

Distance function (squared) is

$$D^2 = x^2 + y^2 + z^2 \quad \text{But } x+y+z=1 \rightarrow z=1-x-y$$

$$D^2 = f(x, y) = x^2 + y^2 + (1-x-y)^2$$

$$\begin{aligned} f_x &= 2x - 2(1-x-y) = 0 \\ f_y &= 2y - 2(1-x-y) = 0 \end{aligned} \quad \left. \begin{array}{l} 2x-2y=0 \\ 2x+2y=2 \end{array} \right\} \rightarrow x=y$$

$$\text{So } z = 1-x-y = 1-x-x = 1-2x \text{ and}$$

$$z^2 = 2x^2 + 2y^2 \rightarrow (1-2x)^2 = 2x^2 + 2x^2$$

$$1-4x+4x^2 = 4x^2$$

$$4x = 1$$

$$x = \frac{1}{4}$$

$$\text{so } y = \frac{1}{4} \quad \therefore \quad z = 1 - 2\left(\frac{1}{4}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$f_{xx} = 4 > 0 \quad f_{yy} = 4 \quad f_{xy} = 2$$

$$D = (4)(4) - (2)^2 = 12 > 0 \quad \text{Local Min}$$

$(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is closest to origin

$$\textcircled{5} \quad \text{Using La Grange} \quad \underbrace{x+y+z=1}_{g(x,y,z)} \quad \underbrace{z^2-2x^2-2y^2=0}_{h(x,y,z)=0}$$

Distance function (squared) is
→ from origin...

$$D^2 = f(x, y, z) = x^2 + y^2 + z^2$$

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle + \mu \langle -4x, -4y, 2z \rangle$$

$$\begin{aligned} 2x &= \lambda - 4\mu x \rightarrow \lambda = 2x(1-2\mu) \\ 2y &= \lambda - 4\mu y \rightarrow \lambda = 2y(1-2\mu) \\ 2z &= \lambda + 2\mu z \\ x+y+z &= 1 \\ z^2 - 2x^2 - 2y^2 &= 0 \end{aligned} \quad \left. \begin{array}{l} 2x-2y=0 \\ x=y \end{array} \right\} \quad \begin{array}{l} \text{OR} \\ 1-2\mu=0 \\ \mu=\frac{1}{2} \end{array}$$

$$\text{Case A} \quad x=y \rightarrow z^2 - 2x^2 - 2x^2 = 0 \quad z^2 = 4x^2 \quad \text{so } z = \pm 2x$$

if $z = -2x$ then $x+y+z = x+x-2x = 0 \neq 1$ NOPE...

$$\text{so } z = 2x \quad \frac{1}{2}x+x+2x=1 \rightarrow x = \frac{1}{4} = y \quad \frac{1}{2}z = \frac{1}{2}$$

Critical Value $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

$$\text{Case B} \quad \mu = \frac{1}{2} \rightarrow \lambda = 0 \rightarrow 2z = \lambda + 2\mu z \text{ becomes } 2z = z \rightarrow z = 0$$

$$\text{so } z^2 - 2x^2 - 2y^2 = 0 \text{ becomes } -2x^2 - 2y^2 = 0 \quad \text{OR}$$

$$x^2 + y^2 = 0 \rightarrow x = y = 0 \quad \text{BUT then } x+y+z = 0 \neq 1$$

NoPE...

Thus our only point $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is the closest to the origin as desired.

$$\textcircled{6} \quad Z = f(x, y) \quad \left\{ \begin{array}{l} x = g(r, \theta) = r \cos \theta \\ y = h(r, \theta) = r \sin \theta \end{array} \right.$$

$$Z_r = Z_x X_r + Z_y Y_r$$

$$Z_r = Z_x \cos \theta + Z_y \sin \theta$$

$$(Z_r)^2 = (Z_x \cos \theta + Z_y \sin \theta)^2$$

A

$$(Z_r)^2 = (Z_x)^2 \cos^2 \theta + 2Z_x Z_y \sin \theta \cos \theta + (Z_y)^2 \sin^2 \theta$$

$$Z_\theta = Z_x X_\theta + Z_y Y_\theta$$

$$Z_\theta = Z_x (-r \sin \theta) + Z_y (r \cos \theta)$$

$$(Z_\theta)^2 = (Z_x)^2 r^2 \sin^2 \theta - 2r^2 Z_x Z_y \sin \theta \cos \theta + (Z_y)^2 r^2 \cos^2 \theta$$

B

$$\frac{1}{r^2} (Z_r)^2 = (Z_x)^2 \sin^2 \theta - 2Z_x Z_y \sin \theta \cos \theta + (Z_y)^2 \cos^2 \theta$$

$$(Z_r)^2 + \frac{1}{r^2} (Z_\theta)^2 = \boxed{A} + \boxed{B} \quad (\text{Note: the green terms add to zero})$$

$$= (Z_x)^2 [\cos^2 \theta + \sin^2 \theta] + (Z_y)^2 [\sin^2 \theta + \cos^2 \theta]$$

$$= (Z_x)^2 + (Z_y)^2 \quad \text{as desired.}$$