Solve each of the following differential equations:

1. $y' + \dfrac{3}{2 - x - x^2} \cdot y = \dfrac{x+1}{2+x}$

2. $\left(x^2+1\right)^{3/2} \cdot \ln y \cdot y' = x^2 y$

3. $y' = \dfrac{xy}{x^2+1}$

4. $x \ln x \cdot y' - \sqrt{1-y^2} = 0$ where $y(e) = 1$

5. $(x+1)\left(x^2+1\right)\dfrac{dy}{dx} = \sqrt{y^2-1}$

6. $\left(x^2+x+1\right)y' = \dfrac{1}{\ln y}$

7. $y' + \tan x \cdot y = \cos^2 x$ where $x \in \left(0, \dfrac{\pi}{2}\right)$ and $y\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

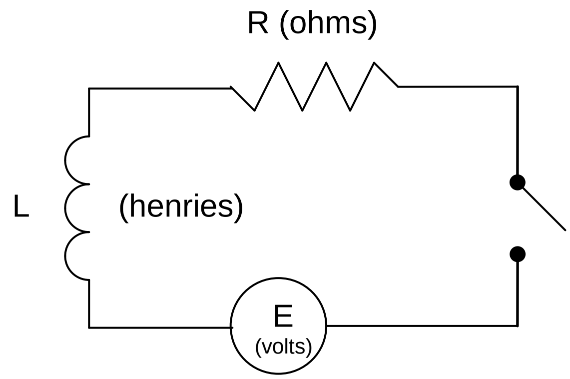
8. $y' = \dfrac{y}{x} + \ln x$

9. $y' + \dfrac{1}{t+1} \cdot y = t + 1$

10. A culture of bacteria is subjected to a controlled environment in order to test a new growth inhibitor. Data from these tests provides a growth model for this bacteria in this particular environment suggesting that the rate of growth is proportional to the tangent function evaluated on the amount of bacteria present at time *t*, but only until the population reaches 1.5 mg. After the population reaches 1.5 mg, it maintains that value. If $\pi/6$ milligrams of this bacteria that is placed in this special environment grows to $\pi/3$ milligrams in 1 hour, then write an appropriate differential equation and solve it. In addition, sketch two graphs: $\dfrac{dP}{dt}$ against *P* and another with *P* against *t* (using technology for this latter graph is fine). When will $\dfrac{dP}{dt}$ become zero?

11. The rate at which a population grows is proportional to the square root of the number of people. If $P(0) = 100$ and $P(1) = 121$, then find a model for this population.

12. Electrifying:



The current I in an RL circuit is mathematically modeled by:

$$L\frac{dI}{dt}+RI=E(t)$$

where L is the inductance on the inductor, measured in henries, R is the resistance on the resistor, measured in ohms, $E(t)$ is the voltage supplied to the system, measured in volts, and I is the current, measured in amps (or amperes). Suppose $L=1$ henry, $R=2$ ohms, and $E(t)=10e^{-2t}$ volts where t is time, measured in seconds. Find I as a function of t if $I(0)=0$. In other words, determine the current as a function of time if the current is initially 0 amps.

13. The rate at which the population of social anarchists grows in the totalitarian regime of *Paraboland* is inversely proportional to the square of the population. If *Paraboland* initially experiences 10 such anarchists, and it will take 3 months for this population to double, then find the population of these anarchists as a function of time.

14. The major water reservoir in the Republic of Flatland holds 100 million gallons supplies the city of Trigland with 1 million gallons a day. The reservoir is partly refilled by a spring that provides 0.9 million gallons a day and the rest, 0.1 million gallons a day, by run-off from the surrounding land. The spring is clean, but the run-off contains salt with a concentration of 0.0001 pound per gallon. Assume that there is no salt in the reservoir initially and that the reservoir is well mixed. Find the concentration of salt in this reservoir as a function of time.

15. $(2x^4+x^3-2x^2+2x-12)\dfrac{dy}{dx}=(2x^5-3x^4+3x^3+10x^2-17x-22)y\sqrt{\ln^2 y+\ln y+1}$

16. A holding-tank has a 200-gallon capacity. Initially, it is only half full with an environmentally hazardous acid solution where the concentration of this acid is $\frac{1}{3}$ lb. of acid per gallon. This holding tank has two hoses pumping fluid in and one hose at the bottom that allows fluid out. The tank, however, has no top and is exposed. Each hose supplying fluid does so at a rate of 6 gal./min. and the hose at the bottom removes fluid at a rate of 7 gal./min. The two supplying

hoses have a concentration of acid which is $\frac{1}{4}$ lb./gal. for one and $\frac{1}{5}$ lb./gal. for the other. Assume that the ever-present mathematical god is present and doing his job with the mixing. Find the concentration of this acid in the tank as a function of time. What is the concentration of this acid when the fluid first starts to overflow the top edge of this tank?

Math 31 Answers or Solutions, Depending on How Much Time I Had

1. $(x+2)y=(x-1)\left(x+\ln(x-1)^2+C\right)$

2. $\ln^2 y=-\frac{x}{\sqrt{x^2+1}}+\ln\frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}-1}+C$

3. $y=ce^{\frac{1}{2}\left(x^2+1\right)}$

4. $y=\cos\left(\ln\left|\ln x\right|\right)$

5. $\ln\left|y+\sqrt{y^2-1}\right|=\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}+\frac{1}{2}\tan^{-1}x+C$

6. $y\ln y-y=\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$

7. $y=\frac{1}{2}\sin 2x+C\cdot\cos x$

8. $y=x\ln^2 x+Cx$

9. $y=\frac{t+1}{2}+\frac{C}{t+1}$

14. $C(t)=\frac{1}{100{,}000}\left(1-e^{-\frac{1}{100}t}\right)$

15. $\left(2x^4+x^3-2x^2+2x-12\right)\frac{dy}{dx}=\left(2x^5-3x^4+3x^3+10x^2-17x-22\right)y\sqrt{\ln^2 y+\ln y+1}$

Separating variables and integrating each side, we have that:

$$\int\frac{2x^5-3x^4+3x^3+10x^2-17x-22}{2x^4+x^3-2x^2+2x-12}dx=\int\frac{1}{y\sqrt{\ln y+\ln y+1}}dy \qquad (1)$$

Let us consider the integral on the left hand side of equation (1) first:

In evaluating $\int \frac{2x^5-3x^4+3x^3+10x^2-17x-22}{2x^4+x^3-2x^2+2x-12}dx$, we will perform polynomial long division to rewrite the improper rational expression in the integrand's position as the sum of a polynomial (the quotient) and a proper rational expression (the ratio of the remainder to the divisor):

$$\begin{array}{r|l} & x\ -\ 2 \\ \hline 2x^4+x^3-2x^2+2x-12 & 2x^5-3x^4+3x^3+10x^2-17x-22 \\ & \underline{2x^5+x^4-2x^3+2x^2-12x} \\ & \quad -4x^4+5x^3+8x^2-5x-22 \\ & \quad \underline{-4x^4-2x^3+4x^2-4x+24} \\ & \qquad 7x^3+4x^2-x-46 \end{array}$$

Consequently, we now have that:

$$\frac{2x^5-3x^4+3x^3+10x^2-17x-22}{2x^4+x^3-2x^2+2x-12}=x-2+\frac{7x^3+4x^2-x-46}{2x^4+x^3-2x^2+2x-12}$$

which allows us to rewrite integral on the left hand side of (1) as follows:

$$\int \frac{2x^5-3x^4+3x^3+10x^2-17x-22}{2x^4+x^3-2x^2+2x-12}dx=\int\left(x-2+\frac{7x^3+4x^2-x-46}{2x^4+x^3-2x^2+2x-12}\right)dx \qquad (2)$$

Next, we will factor the denominator of the proper rational expression. The rational root theorem tells us that if there are any rational zeros, then they must be from among the list:

$r=\frac{\pm1,\pm2,\pm3,\pm4,\pm6,\pm12}{\pm1,\pm2}$, or more consisely, $r=\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm2,\pm3,\pm4,\pm6,\pm12$

Let us consider testing whether, say –2, is a rational zero of that denominator. We will apply synthetic division to determine if this is indeed a zero:

-2	2	1	-2	2	-12
		-4	6	-8	12
	2	-3	4	-6	0

From the integral (2), we now have:

$$\int \frac{2x^5-3x^4+3x^3+10x^2-17x-22}{2x^4+x^3-2x^2+2x-12}dx=\int\left(x-2+\frac{7x^3+4x^2-x-46}{2x^4+x^3-2x^2+2x-12}\right)dx$$

$$= \int \left(x - 2 + \frac{7x^3 + 4x^2 - x - 46}{(x+2)\left(2x^3 - 3x^2 + 4x - 6\right)} \right) dx$$

$$= \int \left(x - 2 + \frac{7x^3 + 4x^2 - x - 46}{(x+2)\left(x^2(2x-3) + 2(2x-3)\right)} \right) dx$$

$$= \int \left(x - 2 + \frac{7x^3 + 4x^2 - x - 46}{(x+2)(2x-3)\left(x^2+2\right)} \right) dx \qquad (3)$$

Next, we will perform partial fraction decomposition to rewrite the proper rational expression in the integral (3) as the sum of more manageable fractions whose denominators are merely linear or quadratic:

$$\frac{7x^3 + 4x^2 - x - 46}{(x+2)(2x-3)\left(x^2+2\right)} = \frac{A}{x+2} + \frac{B}{2x-3} + \frac{Cx+D}{x^2+2}$$

Multiplying both sides of the equation by the LCM of the denominators, we have that:

$$7x^3 + 4x^2 - x - 46 = A(2x-3)\left(x^2+2\right) + B(x+2)\left(x^2+2\right) + (Cx+D)(x+2)(2x-3)$$

To determine the values for A, B, C, and D, we will:

$$\text{let } x = -2:\ 7(-2)^3 + 4(-2)^2 - (-2) - 46 = A(-7)(6)$$

$$-84 = -42A$$

$$A = 2$$

$$\text{let } x = \frac{3}{2}:\ 7\left(\frac{3}{2}\right)^3 + 4\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) - 46 = B \cdot \frac{7}{2} \cdot \frac{17}{4}$$

$$-\frac{119}{8} = B \cdot \frac{119}{8}$$

$$B = -1$$

equating coefficients of x^3, we have that:

$$7 = 2A + B + 2C$$

$$7 = 2 \cdot 2 - 1 + 2C$$

$$C = 2$$

equating constant terms, we have that:

$$-46 = -6A + 4B - 6D$$

$$-46 = -6 \cdot 2 + 4(-1) - 6D$$

$$D = 5$$

Finally, we have from (3) that:

$$\int \frac{2x^5 - 3x^4 + 3x^3 + 10x^2 - 17x - 22}{2x^4 + x^3 - 2x^2 + 2x - 12} dx = \int \left(x - 2 + \frac{7x^3 + 4x^2 - x - 46}{(x+2)(2x-3)\left(x^2+2\right)} \right) dx$$

$$= \int \left(x - 2 + \frac{2}{x+2} - \frac{1}{2x-3} + \frac{2x+5}{x^2+2} \right) dx$$

$$= \int \left(x - 2 + \frac{2}{x+2} - \frac{1}{2x-3} + \frac{2x}{x^2+2} + \frac{5}{x^2+2} \right) dx$$

$$= \frac{1}{2}x^2 - 2x + 2\ln|x+2| - \frac{1}{2}\ln|2x-3| + \ln\left(x^2+2\right) + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

$$= \frac{1}{2}x^2 - 2x + \ln\frac{(x+2)^2\left(x^2+2\right)}{\sqrt{2x-3}} + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Let us next consider the integral on the right hand side of equation (1). We will apply the following substitution:

$$\int \frac{1}{y\sqrt{\ln^2 y + \ln y + 1}} dy \quad \text{where let } u = \ln y$$

$$du = \frac{1}{y} dy$$

$$\int \frac{1}{y\sqrt{\ln^2 y + \ln y + 1}} dy = \int \frac{1}{\sqrt{u^2 + u + 1}} du$$

$$= \int \frac{1}{\sqrt{u^2 + u + \frac{1}{4} + \frac{3}{4}}} du$$

$$= \int \frac{1}{\sqrt{\left(u + \frac{1}{2}\right)^2 + \frac{3}{4}}} du \quad \text{where we will let } u + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$$

$$du = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$$

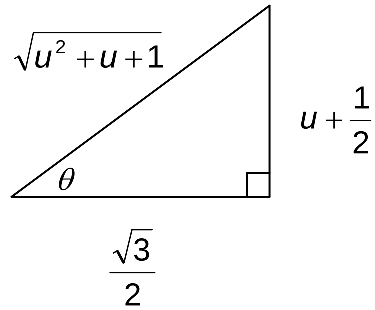
$$= \int \frac{1}{\sqrt{\frac{3}{4}\tan^2\theta + \frac{3}{4}}} \cdot \frac{\sqrt{3}}{2}\sec^2\theta d\theta$$

$$= \int \frac{1}{\sqrt{\frac{3}{4}\tan^2\theta + \frac{3}{4}}} \cdot \frac{\sqrt{3}}{2}\sec^2\theta d\theta$$

$$= \int \frac{1}{\sec\theta}\sec^2\theta d\theta$$

$$= \int \sec\theta d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C_2 \quad \text{where we have the following geometric relationship:}$$



$$
\begin{aligned}
&= \ln\left|\frac{\sqrt{u^2+u+1}}{\sqrt{3}/2} + \frac{u+1/2}{\sqrt{3}/2}\right| + C_2 \\
&= \ln\left|\frac{\sqrt{u^2+u+1}+u+1/2}{\sqrt{3}/2}\right| + C_2 \\
&= \ln\left|\sqrt{u^2+u+1}+u+1/2\right| - \ln\left|\sqrt{3}/2\right| + C_2 \\
&= \ln\left|\sqrt{u^2+u+1}+u+1/2\right| + C_3 \text{ where } C_3 = C_2 - \ln\left|\sqrt{3}/2\right| \\
&= \ln\left|\sqrt{\ln^2 y+\ln y+1}+\ln y+1/2\right| + C_3
\end{aligned}
$$

We now have our solution to the differential equation (1):

$$
\frac{1}{2}x^2 - 2x + \ln\frac{(x+2)^2(x^2+2)}{\sqrt{2x-3}} + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C_1 = \ln\left|\sqrt{\ln^2 y+\ln y+1}+\ln y+1/2\right| + C_3, \text{ or}
$$

$$
\frac{1}{2}x^2 - 2x + \ln\frac{(x+2)^2(x^2+2)}{\sqrt{2x-3}} + \frac{5}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} = \ln\left|\sqrt{\ln^2 y+\ln y+1}+\ln y+1/2\right| + C_4 \text{ with } C_4 = C_3 - C_1
$$

16. A holding-tank has a 200-gallon capacity. Initially, it is only half full with an environmentally hazardous acid solution where the concentration of this acid is $\frac{1}{3}$ lb. of acid per gallon. This holding tank has two hoses pumping fluid in and one hose at the bottom that allows fluid out. The tank, however, has no top and is exposed. Each hose supplying fluid does so at a rate of 6 gal./min. and the hose at the bottom removes fluid at a rate of 7 gal./min. The two supplying hoses have a concentration of acid which is $\frac{1}{4}$ lb./gal. for one and $\frac{1}{5}$ lb./gal. for the other. Assume that the ever-present mathematical god is present and doing his job with the mixing. Find the concentration of this acid in the tank as a function of time. What is the concentration of this acid when the fluid first starts to overflow the top edge of this tank?

Let A=amount of solution in tank, measured in lbs.

V=volume of solution in tank, measured in gallons

C=concentration of acid in tank, measured in lbs/gal., where $C = \frac{A}{V}$

i) Note: V is not constant. After all,

$$\frac{dV}{dt} = 6 + 6 - 7$$

$$\frac{dV}{dt} = 5$$

$$V = 5t + K_1 \text{ where } V(0) = 100$$

$$\text{so, } 100 = 5 \cdot 0 + K_1$$

$$100 = K_1$$

$$V = 5t + 100$$

ii) Next, consider a differential equation involving A:

$$\frac{dA}{dt} = (\text{rate in}) - (\text{rate out})$$

$$\frac{dA}{dt} = (\text{concentration}_{\text{in}})(\text{rate of flow}_{\text{in}}) - (\text{concentration}_{\text{out}})(\text{rate of flow}_{\text{out}})$$

$$\frac{dA}{dt} = \left(\frac{1}{5}\frac{\text{lbs}}{\text{gal.}}\right)\left(6\frac{\text{gal.}}{\text{min.}}\right) + \left(\frac{1}{4}\frac{\text{lbs}}{\text{gal.}}\right)\left(6\frac{\text{gal.}}{\text{min.}}\right) - \left(\frac{A}{V}\frac{\text{lbs}}{\text{gal.}}\right)\left(7\frac{\text{gal.}}{\text{min.}}\right)$$

$$\frac{dA}{dt} = 6 \cdot \frac{9}{20} - \frac{A}{5t + 100} \cdot 7$$

$$\frac{dA}{dt} + \frac{7}{5t + 100} A = \frac{27}{10}$$

This is a first order linear differential equation for which the integrating factor is:

$$I(t) = e^{\int \frac{7}{5t+100} dt}$$

$$= (5t + 100)^{7/5}$$

Consequently, we have that:

$$\frac{d}{dt}\left((5t+100)^{7/5}A\right)=\frac{27}{10}(5t+100)^{7/5}$$

$$(5t+100)^{7/5}A=\frac{27}{10}\cdot\frac{1}{5}\cdot\frac{5}{12}(5t+100)^{12/5}+K_2$$

$$A=\frac{9}{40}(5t+100)+K_2(5t+100)^{-7/5} \text{ where } A(0)=V(0)C(0)$$

$$=100\cdot\frac{1}{3}$$

$$=\frac{100}{3}$$

$$\text{so, we have that: } \frac{100}{3}=\frac{9}{40}\cdot 100+K_2\cdot 100^{-7/5}$$

$$\text{i.e., } K_2=\frac{13}{120}\cdot 100^{12/5}$$

Consequently, we now have that:

$$A(t)=\frac{9}{40}(5t+100)+\frac{13}{120}\cdot 100^{12/5}(5t+100)^{-7/5}$$

So, $C(t)=\dfrac{A(t)}{V(t)}$

$$C(t)=\frac{\frac{9}{40}(5t+100)+\frac{13}{120}\cdot 100^{12/5}(5t+100)^{-7/5}}{5t+100}$$

$$C(t)=\frac{9}{40}+\frac{13}{120}\cdot 100^{12/5}(5t+100)^{-12/5}$$

We next need to find when the tank is full:

Recall that $V=5t+100$ and consequently $V=200$ when $t=20$ hours.

To find the concentration at this point in time, we will evaluate:

$$C(20)=\frac{9}{40}+\frac{13}{120}\cdot 100^{12/5}(5\cdot 20+100)^{-12/5}$$

$$\approx 0.246 \text{ lbs/gal}$$