Provide a clear and organized presentation. Show absolutely all of your work and completely simplify your answer. Evaluate the following integral:

$$\int \frac{6x^4 + 7x^3 - 15x^2 - 8x + 4}{2x^3 + 9x^2 + 7x - 6}\,dx$$

To evaluate $\int \frac{6x^4+7x^3-15x^2-8x+4}{2x^3+9x^2+7x-6}dx$, not that in the integrand's position we have an improper rational expression. To rewrite this improper rational expression as the sum of a polynomial expression with a proper rational expression, let's perform polynomial long division:

$$\begin{array}{r}
3x-10 \\
2x^3+9x^2+7x-6\overline{)6x^4+7x^3-15x^2-8x+4} \\
\underline{6x^4+27x^3+21x^2-18x} \\
-20x^3-36x^2+10x+4 \\
\underline{-20x^3-90x^2-70x+60} \\
54x^2+80x-56
\end{array}$$

Consequently, we now have that:

$$\int \frac{6x^4+7x^3-15x^2-8x+4}{2x^3+9x^2+7x-6}dx = \int\left(3x-10+\frac{54x^2+80x-56}{2x^3+9x^2+7x-6}\right)dx$$

$$= \int(3x-10)dx + \int\frac{54x^2+80x-56}{2x^3+9x^2+7x-6}dx \qquad (*)$$

Now, with this second integral, we have in the integrand's position a proper rational expression. In order to apply partial fractional decomposition, we must first factor the denominator. To perform this factorization, let's apply the Rational Root Theorem. We see that <u>if</u> there are any rational zeros, it will be one from the following list of rational numbers: $\frac{\pm1,\pm2,\pm3,\pm6}{\pm1,\pm2}$, or more simply, $\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm3,\pm6$. Let's try one, say –2:

$$\begin{array}{r|rrrr}
-2 & 2 & 9 & 7 & -6 \\
 & & -4 & -10 & 6 \\
\hline
 & 2 & 5 & -3 & 0
\end{array}$$

This tells us that $2x^3+9x^2+7x-6=(x+2)(2x^2+5x-3)$

$$=(x+2)(x+3)(2x-1)$$

Now we can perform partial fractional decomposition on the proper rational expression within the integrand of the second integral of (*):

$$\frac{54x^2+80x-56}{2x^3+9x^2+7x-6}=\frac{A}{x+2}+\frac{B}{x+3}+\frac{C}{2x-1}$$

$$54x^2+80x-56=A(x+3)(2x-1)+B(x+2)(2x-1)+C(x+2)(x+3)$$

If we let $x=-2$, we have that $216-160-56=A(1)(-5)$, or $A=0$

If we let $x=-3$, we have that $486-240-56=B(-1)(-7)$, or $B=\frac{190}{7}$

If we let $x=\frac{1}{2}$, we have that $\frac{27}{2}+40-56=C\cdot\frac{5}{2}\cdot\frac{7}{2}$, or $C=-\frac{2}{7}$

Finally, we have that:

$$\begin{aligned}\int\frac{6x^4+7x^3-15x^2-8x+4}{2x^3+9x^2+7x-6}dx &= \int\left(3x-10+\frac{54x^2+80x-56}{2x^3+9x^2+7x-6}\right)dx \\ &= \int(3x-10)dx+\int\frac{54x^2+80x-56}{2x^3+9x^2+7x-6}dx \\ &= \int(3x-10)dx+\int\left(\frac{190/7}{x+3}+\frac{-2/7}{2x-1}\right)dx \\ &= \frac{3}{2}x^2-10x+\frac{190}{7}\ln|x+3|-\frac{2}{7}\cdot\frac{1}{2}\ln|2x-1|+C \\ &= \frac{3}{2}x^2-10x+\frac{1}{7}\ln\frac{(x+3)^{190}}{|2x-1|}+C\end{aligned}$$