Give exact values only and do not use a calculator for any part of this exam, unless otherwise specified. Completely simplify your answers, be clear and organized, and show all of your work. Nowhere on this exam shall we employ the Growth Rate Theorem.

1. (12 pts) Determine the area bounded by the graphs of the equations:

$$y = \tanh x,\ y = 1,\ \text{and } x = 0$$

$$
\begin{aligned}
A &= \int_0^\infty (1 - \tanh x)\,dx \\
&= \lim_{t\to\infty} \int_0^t (1 - \tanh x)\,dx \\
&= \lim_{t\to\infty} \int_0^t \left(1 - \frac{\sinh x}{\cosh x}\right) dx \\
&= \lim_{t\to\infty} \int_0^t \left(1 - \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}}\right) dx \\
&= \lim_{t\to\infty} \int_0^t \left(1 - \frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx \\
&= \lim_{t\to\infty} \int_0^t \frac{2e^{-x}}{e^x + e^{-x}}\,dx \\
&= \lim_{t\to\infty} \int_0^t \frac{2e^{x}}{e^{3x} + e^{x}}\,dx \\
&= \lim_{t\to\infty} \int_1^{e^t} \frac{2}{u^3 + u}\,du \\
&= \lim_{t\to\infty} \int_1^{e^t} \frac{2}{u(u^2 + 1)}\,du \\
&= \lim_{t\to\infty} \int_1^{e^t} \left(\frac{A}{u} + \frac{Bu + C}{u^2 + 1}\right) du \\
&= \lim_{t\to\infty} \int_1^{e^t} \left(\frac{2}{u} + \frac{-2u}{u^2 + 1}\right) du \\
&= \lim_{t\to\infty} \left(2\ln u - \ln(u^2 + 1)\right)\Big|_1^{e^t} \\
&= \lim_{t\to\infty} \left(\ln\frac{e^{2t}}{e^{2t} + 1} - \ln\frac{1}{2}\right) \\
&= -\ln\frac{1}{2} \\
&= \ln 2
\end{aligned}
$$

2. (24 pts) Solve the differential equation $\left(2x^3 - x^2 + x - 6\right)\frac{dy}{dx} = \sec^4 y$

$$\int \cos^4 y\, dy = \int \frac{1}{2x^3 - x^2 + x - 6} dx$$

$$\int \left(\cos^2 y\right)^2 dy = \int \frac{1}{(2x-3)\left(x^2 + x + 2\right)} dx$$

$$\int \left(\frac{1+\cos 2y}{2}\right)^2 dy = \int \left(\frac{A}{2x-3} + \frac{Bx + C}{x^2 + x + 2}\right) dx$$

$$\frac{1}{4}\int \left(1 + 2\cos 2y + \cos^2 2y\right) dy = \int \left(\frac{\frac{4}{23}}{2x-3} + \frac{-\frac{2}{23}x - \frac{5}{23}}{x^2 + x + 2}\right) dx$$

$$\frac{1}{4}\int \left(1 + 2\cos 2y + \frac{1+\cos 4y}{2}\right) dy = \int \left(\frac{\frac{4}{23}}{2x-3} + \frac{-\frac{2}{23}x - \frac{1}{23}}{x^2 + x + 2} + \frac{-\frac{4}{23}}{x^2 + x + 2}\right) dx$$

$$\frac{1}{8}\int \left(3 + 4\cos 2y + \cos 4y\right) dy = \int \left(\frac{\frac{4}{23}}{2x-3} + \frac{-\frac{2}{23}x - \frac{1}{23}}{x^2 + x + 2} + \frac{-\frac{4}{23}}{\left(x + \frac{1}{2}\right)^2 + \frac{7}{4}}\right) dx$$

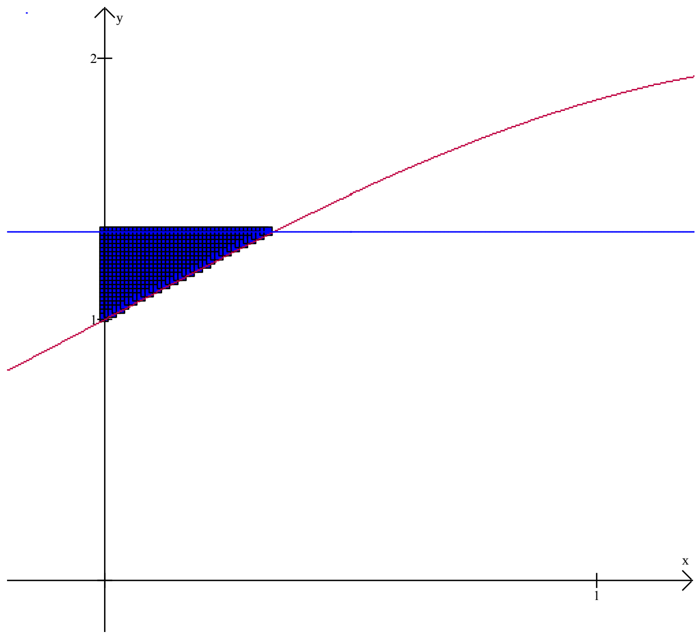
$$\frac{1}{8}\left(3y + 2\sin 2y + \frac{1}{4}\sin 4y\right) = \frac{2}{23}\ln\left|2x-3\right| - \frac{1}{23}\ln\left(x^2 + x + 2\right) - \frac{4}{23}\cdot\frac{2}{\sqrt{7}}\tan^{-1}\frac{2x+1}{\sqrt{7}} + C$$

$$\frac{1}{32}\left(12y + 8\sin 2y + \sin 4y\right) = \frac{1}{23}\ln\frac{(2x-3)^2}{x^2 + x + 2} - \frac{8}{23\sqrt{7}}\tan^{-1}\frac{2x+1}{\sqrt{7}} + C$$

$$12y + 8\sin 2y + \sin 4y = \frac{32}{23}\ln\frac{(2x-3)^2}{x^2 + x + 2} - \frac{8}{23\sqrt{7}}\tan^{-1}\frac{2x+1}{\sqrt{7}} + C$$

3. (12 pts) Let $f(x) = 1 + \sin x$ where $x \in \left[0, \frac{\pi}{2}\right]$. Determine the volume of the solid of revolution obtained by revolving the region bounded by the following equations about the line $x = \pi$:

$$y = f(x), y = \frac{4}{3}, \text{ and } x = 0$$



$$V = 2\pi \int_0^{\sin^{-1}\frac{1}{3}} (\pi - x)\left(\frac{4}{3} - (1 + \sin x)\right) dx$$

$$= 2\pi \int_0^{\sin^{-1}\frac{1}{3}} (\pi - x)\left(\frac{1}{3} - \sin x\right) dx$$

$$= 2\pi \int_0^{\sin^{-1}\frac{1}{3}} \left(\frac{\pi}{3} - \pi \sin x - \frac{1}{3}x + x \sin x\right) dx$$

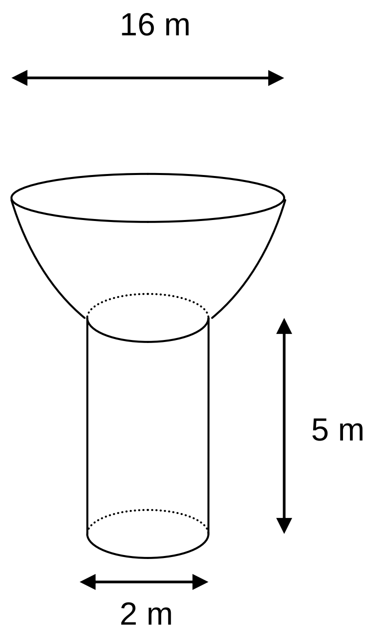
$$= 2\pi \left(\frac{\pi}{3}x + \pi \cos x - \frac{1}{6}x^2 - x\cos x + \sin x\right)\Bigg|_0^{\sin^{-1}\frac{1}{3}}$$

$$= 2\pi \left(\frac{\pi}{3}\sin^{-1}\frac{1}{3} + \pi\frac{2\sqrt{2}}{3} - \frac{1}{6}\left(\sin^{-1}\frac{1}{3}\right)^2 - \frac{2\sqrt{2}}{3}\sin^{-1}\frac{1}{3} + \frac{1}{3} - 0 - \pi + 0 + 0 - 0\right)$$

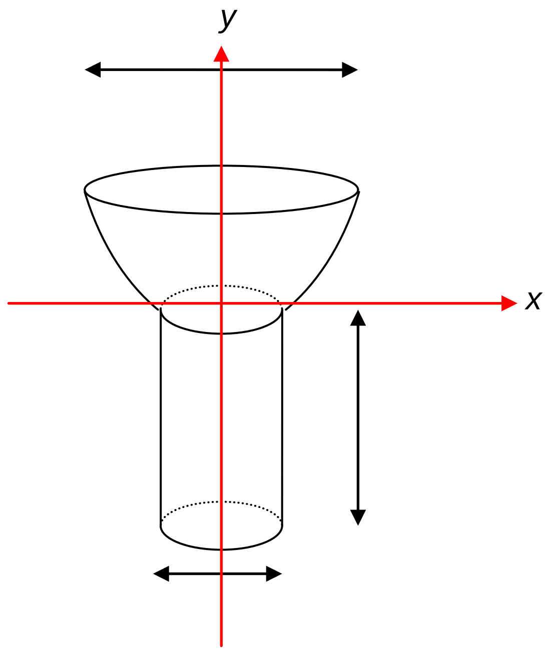
$$= 2\pi \left(\frac{\pi}{3}\sin^{-1}\frac{1}{3} + \pi\frac{2\sqrt{2}}{3} - \frac{1}{6}\left(\sin^{-1}\frac{1}{3}\right)^2 - \frac{2\sqrt{2}}{3}\sin^{-1}\frac{1}{3} + \frac{1}{3} - \pi\right)$$

$$= \frac{\pi}{3}\left(2\pi \sin^{-1}\frac{1}{3} + 4\sqrt{2}\pi - \left(\sin^{-1}\frac{1}{3}\right)^2 - 4\sqrt{2}\sin^{-1}\frac{1}{3} + 2 - 6\pi\right)$$

4. (12 pts) The outer boundary of the upper portion of the tank pictured below is that of a parabola whose focus is ¼ m from the vertex. If the tank is full of water and there is a spout (not pictured) that is 3 m above the tank, then determine how much work is required to pump the water out of the tank. We need only set up the integral: we need not evaluate the integral.



If the parabola has a vertex at the origin (we will change this in a moment), then its equation is $x^2 = 4py$, or in our case, $x^2 = y$. But we will place our coordinate axis system as follows:



which means that the equation of our parabola now becomes $y = x^2 - 1$

Now we have for the parabolic portion:

- $$v_i = \pi \cdot x_i^2 \cdot \Delta y = \pi\left(y_i + 1\right)\Delta y$$
- $$f_i = \rho g v_i = \rho g \pi\left(y_i + 1\right)\Delta y$$
- $$w_i = f_i d_i = \rho g \pi\left(y_i + 1\right)\Delta y\left(66 - y_i\right)$$

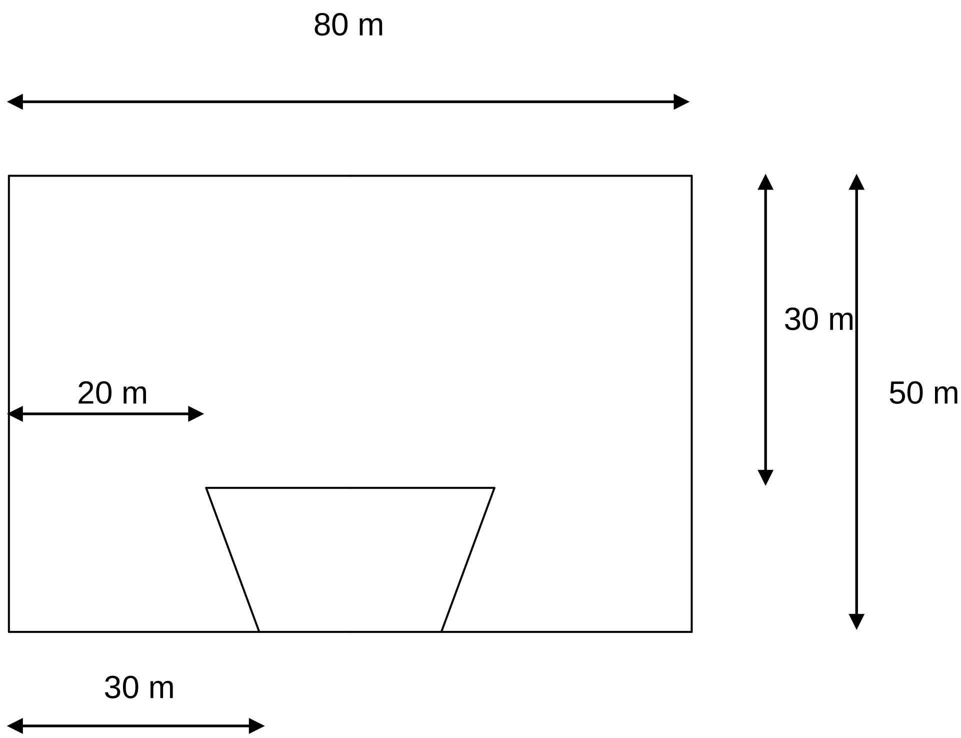
and for the cylindrical portion:

- $$v_i = \pi \cdot 1^2 \cdot \Delta y = \pi \cdot \Delta y$$
- $$f_i = \rho g v_i = \rho g \pi \Delta y$$
- $$w_i = f_i d_i = \rho g \pi \Delta y\left(66 - y_i\right)$$

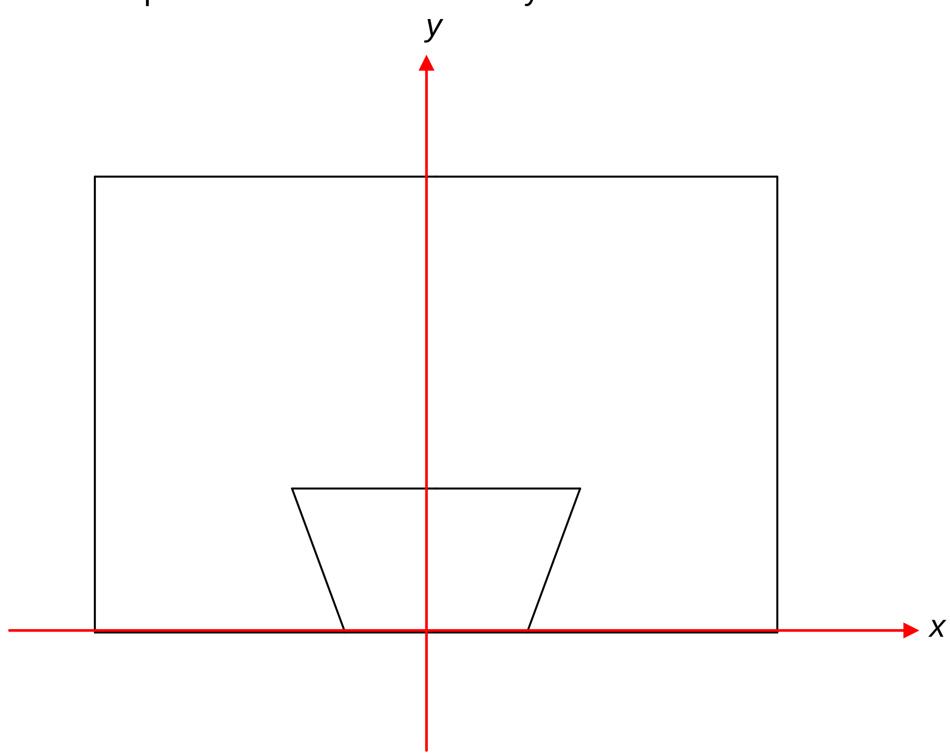
Giving us the integrals:

$$W = 9800\pi\left(\int_0^{63}\left(y+1\right)\left(66-y\right)dy + \int_{-5}^{0}\left(66-y\right)dy\right)$$ Newton-meters, or joules

5. (10 pts) There is a wall at a local aquarium that has dimensions 80 m by 50 m, but has a viewing trapezoidal window at the bottom as indicated in the following picture. How much force is exerted on this viewing window due to hydrostatic pressure? Assume symmetry with respect to a vertical line of symmetry and assume the tank is full of water.



If we place our coordinate axis system a such:



then the boundary curve (or line in this case) for the edge of the window passes through the two points $(10,0)$ and $(20,20)$, which means that the equation of that line is merely $y-y_1=m(x-x_1)$, or $y-0=2(x-10)$, or better yet, $y=2x-20$. Now we see that:

- $$\begin{aligned} A_i &= 2\cdot x_i\cdot\Delta y \\ &= 2\cdot\frac{y_i+20}{2}\cdot\Delta y \\ &= (y_i+20)\Delta y \end{aligned}$$

- $$\begin{aligned} p_i &= \rho g d_i \\ &= \rho g(50-y_i) \end{aligned}$$

- $$\begin{aligned} f_i &= p_i A_i \\ &= \rho g(50-y_i)(y_i+20)\Delta y \end{aligned}$$

This gives us the total force due to hydrostatic pressure to be:

$$\begin{aligned} F &= 9800\int_0^{20}(50-y)(y+20)\,dy \\ &= 9800\int_0^{20}\left(1000+30y-y^2\right)dy \\ &= 9800\left(1000y+15y^2-\frac{y^3}{3}\right)\Bigg|_0^{20} \\ &= 9800\left(1000\cdot 20+15\cdot 400-\frac{8000}{3}\right) \\ &= 9800\left(20{,}000+6{,}000-\frac{8{,}000}{3}\right) \\ &= 9800\cdot\frac{70{,}000}{3} \\ &= \frac{686{,}000{,}000}{3}\text{ Newtons} \end{aligned}$$