Show all of your work, be clear and organized, and nowhere on this exam shall we employ the Growth Rate Theorem.

1. Determine the first four terms of the Taylor Series for $f(x)=\frac{1}{x\sqrt{x}}$ centered about $x=4$.

$$f(x)=x^{-3/2}=\frac{1}{x\sqrt{x}} \text{ and } f(4)=\frac{1}{8}$$

$$f'(x)=-\frac{3}{2}x^{-5/2}=-\frac{3}{2}\cdot\frac{1}{x^2\sqrt{x}} \text{ and } f'(4)=\frac{-3}{64}$$

$$f''(x)=\frac{15}{4}x^{-7/2}=\frac{15}{4}\cdot\frac{1}{x^3\sqrt{x}} \text{ and } f'(4)=\frac{15}{512}$$

$$f'''(x)=-\frac{105}{8}x^{-9/2}=-\frac{105}{8}\cdot\frac{1}{x^4\sqrt{x}} \text{ and } f'''(4)=\frac{-105}{4096}$$

$$f(x)=f(4)+f'(4)(x-4)+\frac{f''(4)}{2!}(x-4)^2+\frac{f'''(4)}{3!}(x-4)^3$$

$$f(x)=\frac{1}{8}-\frac{3}{64}(x-4)+\frac{15}{1024}(x-4)^2-\frac{35}{8192}(x-3)^3$$

2. Using what we know about an infinite geometric series, determine a power series representation for $f(x)=\ln\left(2+x^2\right)$ both in expanded form and using our summation notation.

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

$$\frac{1}{1-\frac{x}{2}}=1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$$

$$2\cdot\frac{1}{2-x}=1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots$$

$$2\cdot\frac{1}{2+x^2}=1-\frac{1}{2}x^2+\frac{1}{4}x^4-\frac{1}{8}x^6+\cdots$$

$$\frac{2x}{2+x^2}=x-\frac{1}{2}x^3+\frac{1}{4}x^5-\frac{1}{8}x^7+\cdots$$

$$\ln\left(2+x^2\right)=\int\left(x-\frac{1}{2}x^3+\frac{1}{4}x^5-\frac{1}{8}x^7+\cdots\right)dx$$

$$\ln\left(2+x^2\right)=C+\frac{x^2}{2}-\frac{x^4}{2\cdot2\cdot2}+\frac{x^6}{2\cdot2\cdot2\cdot3}-\frac{1}{2\cdot2\cdot2\cdot2\cdot4}x^7+\cdots$$

$$\ln\left(2+x^2\right)=\ln2+\frac{x^2}{2}-\frac{x^4}{2\cdot2\cdot2}+\frac{x^6}{2\cdot2\cdot2\cdot3}-\frac{1}{2\cdot2\cdot2\cdot2\cdot4}x^7+\cdots$$

$$\ln\left(2+x^2\right)=\ln2+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{24}-\frac{1}{64}x^7+\cdots$$

$$\ln\left(2+x^2\right)=\ln2+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot2^n}\cdot x^{2n}$$

3. Determine the radius of convergence for the following power series:

$$\sum_{n=1}^{\infty} \frac{(-3)^n (n+3)(2n!)}{n^{2n}} \cdot (x-2)^n$$

According to the Ratio Test, we have convergence for the values on x satisfying:

$$\lim_{n\to\infty} \frac{3^{n+1}(n+4)(2(n+1)!)}{(n+1)^{2(n+1)}} \cdot \frac{n^{2n}}{3^n (n+3)(2n!)} \cdot |x-2| < 1$$

$$\lim_{n\to\infty} \frac{3^{n+1}(n+4)(2n+2)(2n+1)(2n)!}{(n+1)^2 3^n (n+3)(2n!)} \cdot \frac{n^{2n}}{(n+1)^{2n}} \cdot |x-2| < 1$$

$$\lim_{n\to\infty} 3 \cdot \lim_{n\to\infty} \frac{(n+4)(2n+2)(2n+1)}{(n+1)^2 (n+3)} \cdot \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} \cdot |x-2| < 1$$

$$3 \cdot 4 \cdot \frac{1}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{2n}} \cdot |x-2| < 1$$

$$12 \cdot \frac{1}{\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}\right)^2} \cdot |x-2| < 1$$

$$\frac{12}{e^2} \cdot |x-2| < 1$$

$$|x-2| < \frac{e^2}{12}$$

So, the radius of convergence is $\frac{e^2}{12}$

4. Determine whether the series converges or diverges and check any convergence with absoluteness.

i) $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

Clearly, $a_n=\frac{1}{n\ln n}>\frac{1}{(n+1)\ln(n+1)}=a_{n+1}$, so the underlying sequence $\{a_n\}$ where $a_n=\frac{1}{n\ln n}$ is decreasing. The other prerequisites for the Integral Test are clearly met. In addition,

$$\int_2^{\infty}\frac{1}{x\ln x}dx=\lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}dx=\lim_{t\to\infty}\left(\ln(\ln t)-\ln(\ln 2)\right)=\infty.$$

Since this improper integral diverges, then so does our series via the Integral Test.

ii) $\sum_{n=1}^{\infty}\frac{1}{3^n-\sqrt{n}}$ Note that $\lim_{n\to\infty}\frac{3^n}{3^n-\sqrt{n}}=1>0$ and $\sum\frac{1}{3^n}$ converges as a geometric series with $|r|=\frac{1}{3^n}<1$. So, by the Limit Comparison Test, our series converges as well.

iii) $\sum_{n=1}^{\infty}(-1)^n\frac{2^{1/n}}{n!}$

Consider

$$\lim_{n\to\infty}\frac{2^{1/n+1}}{(n+1)!}\cdot\frac{n!}{2^{1/n}}=\lim_{n\to\infty}\frac{2^{1/n+1}}{2^{1/n}(n+1)}=\frac{\lim_{n\to\infty}2^{1/n+1}}{\lim_{n\to\infty}2^{1/n}\lim_{n\to\infty}(n+1)}=\frac{1}{1\cdot\lim_{n\to\infty}(n+1)}=0<1,$$

so our series converges absolutely by the ratio test.

iv) $\sum_{n=1}^{\infty}(-1)^n\frac{n^2+1}{n^3+1}$

Note that if $f(x)=\frac{x^2+1}{x^3+1}$, then $f'(x)=\frac{(x^3+1)\cdot 2x-3x^2\cdot(x^2+1)}{(x^3+1)^2}$

$=\frac{-x^4-3x^2+2x}{(x^3+1)^2}$ which is clearly negative for all values of $x>1$. It is also clear that $\lim_{n\to\infty}\frac{n^2+1}{n^3+1}=0$. So, by the Alternating Series Test, our series converges. Note, however, that $\lim_{n\to\infty}\frac{\frac{n^2+1}{n^3+1}}{\frac{1}{n}}=1>0$ and that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges as the Harmonic Series. So, by the Limit Comparison Test, $\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+1}$ diverges as well and therefore $\sum_{n=1}^{\infty}(-1)^n\frac{n^2+1}{n^3+1}$ converges conditionally.

v) $\sum_{n=1}^{\infty}(-1)^n\frac{n^2+1}{3n^2-2}$. Note that $\lim_{n\to\infty}(-1)^n\frac{n^2+1}{3n^2-2}$ does not evaluate to a single real number since $\frac{n^2+1}{3n^2-2}$ oscillates repeatedly between positive and negative values that themselves approach $\frac{1}{3}$ and $-\frac{1}{3}$.

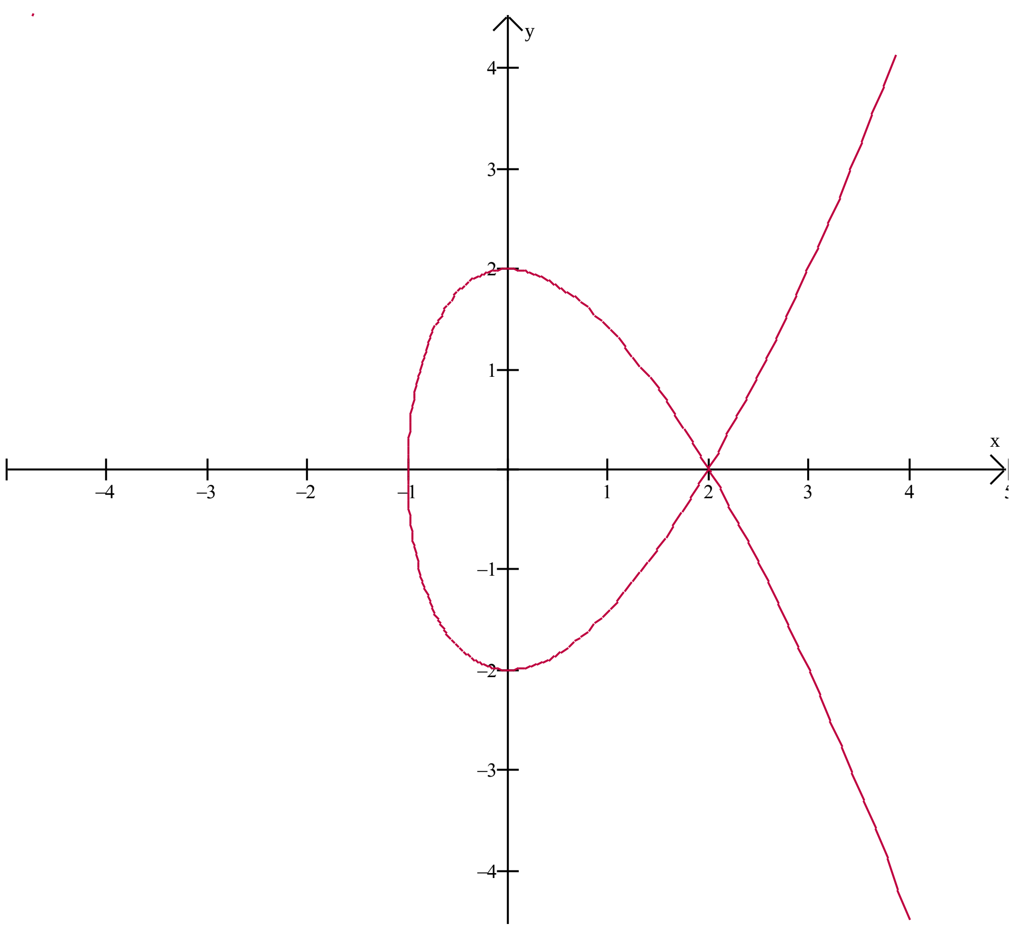
5. Use our *analysis of intervals* technique to sketch the graph of the plane curve described by the following parametric equations:

$$x = t^2 - 1$$
$$y = t^3 - 3t$$

Note that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$. So, at $t = 0$, our tangent line is vertical and at $t = \pm 1$, our tangent line is horizontal.

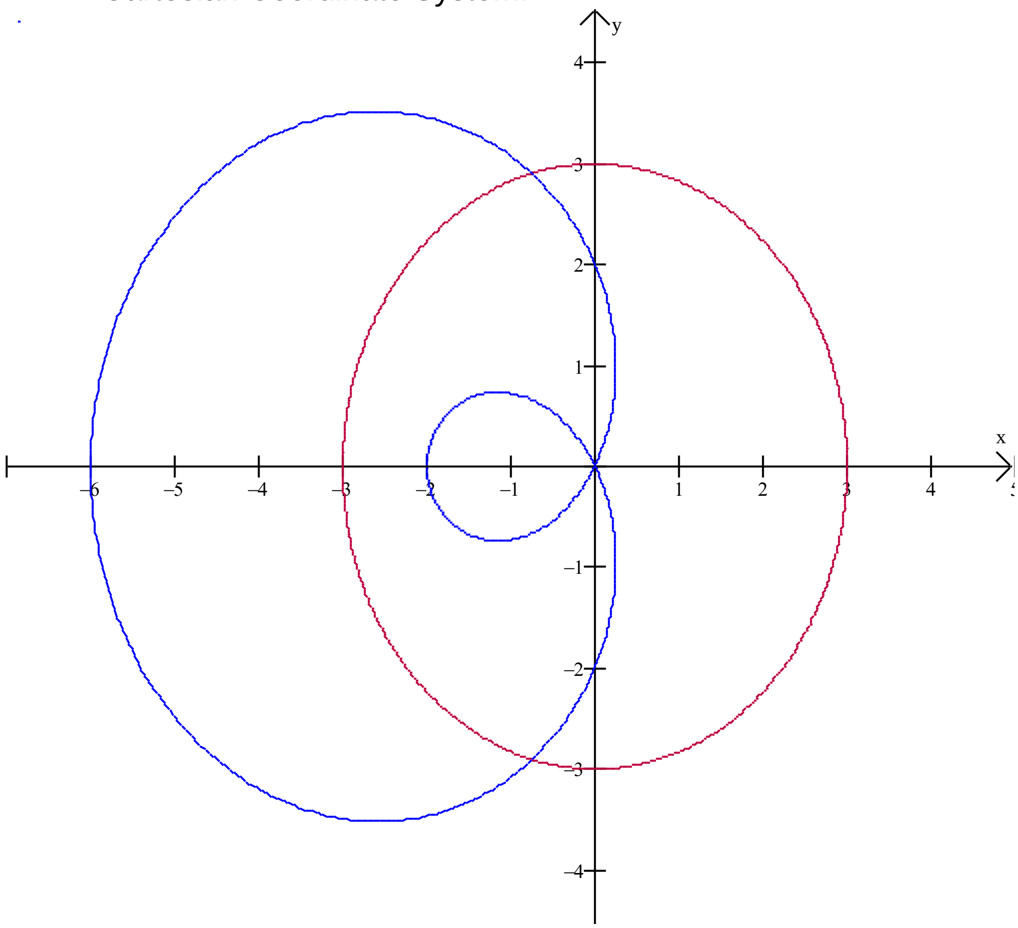
t	$\frac{dx}{dt}$	$\frac{dy}{dt}$	x	y	Curve
$t < -1$	-	+	$\leftarrow$	$\uparrow$	□
$-1 < t < 0$	-	-	$\leftarrow$	$\downarrow$	□
$0 < t < 1$	+	-	$\rightarrow$	$\downarrow$	□
t>1	+	+	$\rightarrow$	$\uparrow$	□

This give us:



7. Consider the equations $r=2-4\cos\theta$ and $r=3$.

i) Clearly sketch the polar graphs of these two equations in the same Cartesian Coordinate System:



ii) Determine the area inside the graph of $r=3$ but outside the graph of $r=2-4\cos\theta$.

$$A=2\cdot\frac{1}{2}\int_0^{\cos^{-1}\left(-\frac{1}{4}\right)}9\,d\theta-2\cdot\frac{1}{2}\int_{\pi/3}^{\cos^{-1}\left(-\frac{1}{4}\right)}(2-4\cos\theta)^2\,d\theta$$
$$=-3\cos^{-1}\left(-\frac{1}{4}\right)+\frac{9}{2}\sqrt{15}+4\pi-6\sqrt{3}$$

iii) At what values of θ in the interval $[0,2\pi]$ is $\frac{dy}{dx}=0$?

Solving $\frac{dr}{d\theta}\sin\theta+r\cos\theta=0$, or $4\sin^2\theta+2\cos\theta-4\cos^2\theta=0$, we have the following solutions:

$$\left\{\cos^{-1}\frac{1+\sqrt{33}}{8},\cos^{-1}\frac{1-\sqrt{33}}{8},2\pi-\cos^{-1}\frac{1+\sqrt{33}}{8},2\pi-\cos^{-1}\frac{1-\sqrt{33}}{8}\right\}$$

Now, truly, wasn't that an absolute piece of cake?