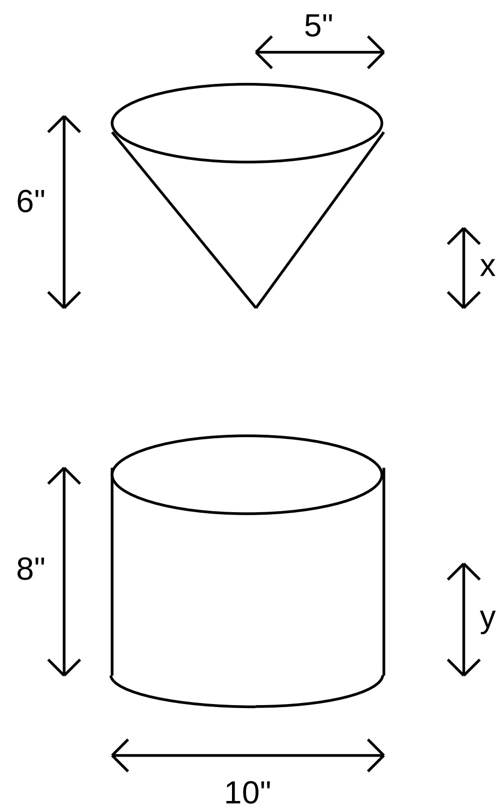
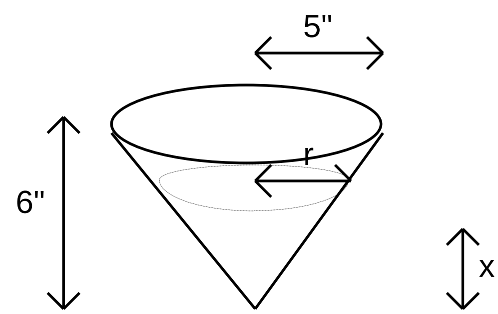
Water from a conical filter drips into a cup that is in the shape of a right circular cylinder. The dimensions of the cone and cup are given in the picture below. Let x represent the depth of the water in the filter and y the depth of the water in the cup. If $30\pi \text{ in}^3$ of water is poured into the filter and drips out of the filter at a rate of $3 \text{ in}^3/\text{min.}$, then how fast is the water level in the cone changing when $x = 1$ in.? How fast is the water level in the cup changing when $x = 1$ in.? What is the depth of the water in the cup when $x = 1$ in.? Give exact values first, then approximate to the nearest 0.01.

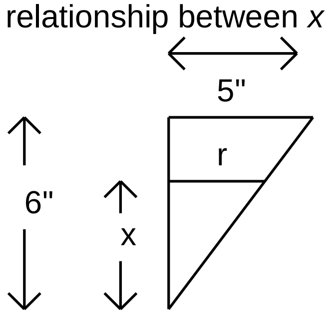




- Find $\frac{dx}{dt}$: Relating the quantities *V*, *r*, and *x* in the cone, we have that:

$$V = \frac{1}{3}\pi r^2 x \qquad (1)$$

Note that if we treat *V*, *r*, and *x* as functions of time *t*, then to differentiate the right-hand side w.r.t. *t*, we must employ a combination of the product rule and the chain rule. In addition, we will have obtained an equation in $\frac{dV}{dt}, \frac{dr}{dt}, \frac{dx}{dt}, r,$ and x. To avoid this, let's rewrite *V* solely in terms of *x*, and then differentiate implicitly with respect to *t*. To accomplish this, consider the following relationship between *x* and *r* due to our having two triangles that are similar:



By similar triangles, we have that $\frac{r}{5} = \frac{x}{6}$

$$\text{i.e., } r = \frac{5}{6}x$$

Substituting this expression in *x* for *r* into equation (1), we have that:

$$V = \frac{1}{3}\pi\left(\frac{5}{6}x\right)^2 x$$

$$V = \frac{25}{108}\pi x^3$$

Now, Treating both *V* and *x* as functions of time *t* and differentiating w.r.t. *t*, we see that:

$$\frac{dV}{dt} = \frac{75}{108}\pi x^2 \frac{dx}{dt}$$

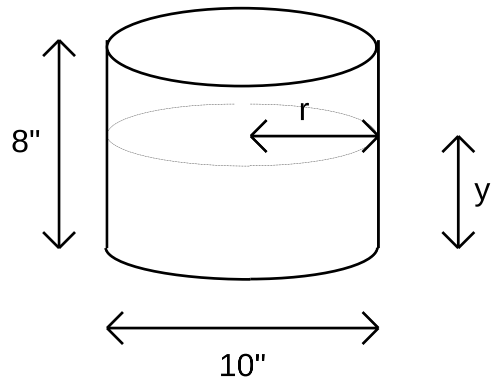
Substituting into this equation values known when $x = 1\text{in.}$, we have that:

$$-3 = \frac{75}{108}\pi 1^2 \frac{dx}{dt}$$

$$\frac{dx}{dt} = -3 \cdot \frac{108}{75} \cdot \frac{1}{\pi}$$

$$= -\frac{108}{25\pi} \text{ in/min.}$$

$$\approx -1.38 \text{ in/min.}$$



- Next, we need to find $\frac{dy}{dt}$: Relating the quantities *V*, *r*, and *y* in the right circular cylinder, note that *r* is fixed at 5 inches. So, we have that:

$$V = \pi r^2 y$$
$$V = \pi \cdot 25 \cdot y$$
$$V = 25\pi y$$

Treating both *V* and *y* as functions in time *t* and differentiating with respect to *t*, we have:

$$\frac{dV}{dt} = 25\pi \frac{dy}{dt}$$
$$3 = 25\pi \frac{dy}{dt}$$
$$\frac{dy}{dt} = \frac{3}{25\pi} \text{ in/min.}$$
$$\approx 0.04 \text{ in/min.}$$

- Finally, to find the depth of the water *y* when $x = 1$ in., note that $V_{\text{cylinder}} = 30\pi - V_{\text{cone}}$. Now, when $x = 1$ in., we have that:

$$V_{\text{cylinder}} = 30\pi - \frac{1}{3}\pi \cdot \left(\frac{5}{6}\right)^2 \cdot 1$$
$$= 30\pi - \frac{25}{108}\pi$$
$$= \frac{3215}{108}\pi$$

Consequently, we have that:

$$y = \frac{V_{\text{cylinder}}}{\pi r^2}$$
$$= \frac{\frac{3215}{108}\pi}{\pi \cdot 5^2}$$
$$= \frac{643}{540} \text{ in.}$$
$$\approx 1.19 \text{ in.}$$