

1. Use the ε, δ definition of the limit to prove $\lim_{x \rightarrow 5} (2x - 3) = 7$

- Part 1: Find a δ that works:

Given $\varepsilon > 0$, we need to find a δ such that:

$$\begin{array}{ll} |2x - 3 - 7| < \varepsilon & \text{whenever } 0 < |x - 5| < \delta \\ |2x - 10| < \varepsilon & \text{whenever } " \\ |2(x - 5)| < \varepsilon & \text{whenever } " \\ 2|x - 5| < \varepsilon & \text{whenever } " \\ |x - 5| < \varepsilon/2 & \text{whenever } " \end{array}$$

So, choose $\delta = \varepsilon/2$

- Part 2: Show that this δ works:

Given $\varepsilon > 0$, choose $\delta = \varepsilon/2$

$$\begin{aligned} \text{If } 0 < |x - 5| < \delta, \text{ then } |2x - 3 - 7| &= |2x - 10| \\ &= |2(x - 5)| \\ &= 2|x - 5| \\ &< 2 \cdot \delta \\ &= 2 \cdot \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

$\therefore \lim_{x \rightarrow 5} (2x - 3) = 7$

2. Prove that $\lim_{x \rightarrow 3} (x^2 + 2x - 5) = 10$:

Part I: Find a δ that works:

Given an $\varepsilon > 0$, we need a $\delta > 0$ such that

$$|(x^2 + 2x - 5) - 10| < \varepsilon \text{ whenever } 0 < |x - 3| < \delta$$

$$\text{But } |(x^2 + 2x - 5) - 10| < \varepsilon \text{ whenever } 0 < |x - 3| < \delta$$

is equivalent to:

$$|x^2 + 2x - 15| < \varepsilon \quad \text{whenever } 0 < |x - 3| < \delta$$

$$|(x + 5)(x - 3)| < \varepsilon \quad \text{whenever } 0 < |x - 3| < \delta$$

$$|x + 5| \cdot |x - 3| < \varepsilon \quad \text{whenever } 0 < |x - 3| < \delta$$

If we can find some number C such that $|x + 5| < C$, then we will have that

$$|x - 3| < \frac{\varepsilon}{C}, \text{ and we will choose } \delta = \frac{\varepsilon}{C}.$$

Aside: Find C :

A reasonable choice for δ is 1. If we choose $\delta = 1$, then we have that:

$$0 < |x - 3| < 1$$

$$-1 < x - 3 < 1$$

$$7 < x + 5 < 9$$

$$|x + 5| < 9$$

So, we now have a value for C to be 9. Thus, an alternative to 1 as a choice for δ is $\frac{\varepsilon}{9}$.

Now that we have two values for δ , to ensure that we obtain the desired tolerance value ε , we will choose $\delta = \min\{1, \frac{\varepsilon}{9}\}$.

Part II: Show that our choice for δ works:

Given $\varepsilon > 0$, choose $\delta = \min\{1, \frac{\varepsilon}{9}\}$.

$$\begin{aligned} \text{If } 0 < |x - 3| < \delta, \text{ then } |(x^2 + 2x - 5) - 10| &= |x^2 + 2x - 15| \\ &= |(x + 5)(x - 3)| \\ &= |x + 5| \cdot |x - 3| \end{aligned}$$

Note: If $\delta = 1$, then $0 < |x - 3| < 1$

which implies $-1 < x - 3 < 1$

i.e., $7 < x + 5 < 9$

or $|x + 5| < 9$

But, if $\delta = \frac{\varepsilon}{9}$, then we have that $0 < |x - 3| < \frac{\varepsilon}{9}$

$$\begin{aligned} \text{Now we have that } |(x^2 + 2x - 5) - 10| &= |x + 5| \cdot |x - 3| \\ &< 9 \cdot \frac{\varepsilon}{9} \\ &= \varepsilon \end{aligned}$$

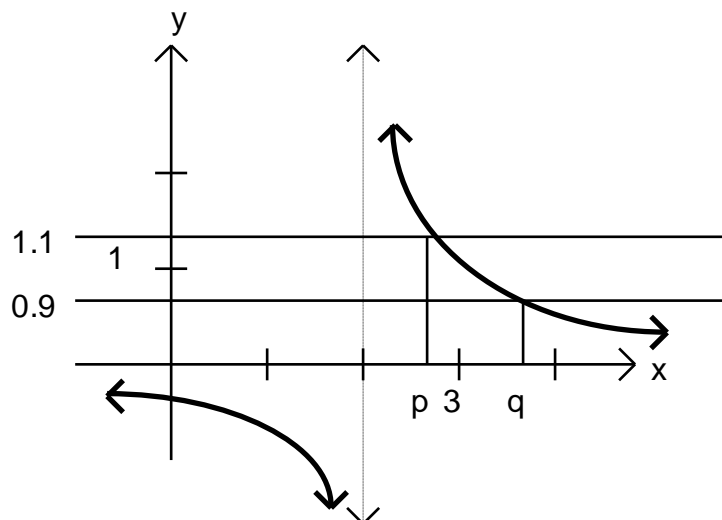
We have shown that if $0 < |x - 3| < \delta$, then $|(x^2 + 2x - 5) - 10| < \varepsilon$

$$\therefore \lim_{x \rightarrow 3} (x^2 + 2x - 5) = 10$$

3. In answering the following problem, round all values to the nearest 0.01.

Given $f(x) = \frac{1}{x-2}$, $\lim_{x \rightarrow 3} f(x) = 1$, and $\varepsilon = 0.1$, find the largest value of δ such that

If $0 < |x-3| < \delta$, then $|f(x)-1| < \varepsilon$. Draw a picture to support your claim.



- Note: To find the values of p and q in the above picture, solve the equations:

$$\frac{1}{p-2} = 1.1$$

$$\frac{1}{q-2} = 0.9$$

$$\frac{1}{1.1} = p-2$$

$$\frac{1}{0.9} = q-2$$

$$p = 2 + \frac{1}{1.1}$$

$$q = 2 + \frac{1}{0.9}$$

$$p \approx 2.91$$

$$q \approx 3.11$$

- Choose δ as follows:

$$\delta = \min\{|3-p|, |3-q|\}$$

$$= \min\{0.09, 0.11\}$$

$$= 0.09$$

Consider that the above suggests that if a value of x is chosen within 0.09 units from 3, $f(x)$ is guaranteed to be within 0.1 units of 1.

4. Use the ε, δ definition of the limit to prove $\lim_{x \rightarrow 5} (3x - 4) = 11$

- Part 1: Find a δ that works:

Given $\varepsilon > 0$, we need to find a δ such that:

$$\begin{aligned} |3x - 4 - 11| < \varepsilon & \text{ whenever } 0 < |x - 5| < \delta \\ |3x - 15| < \varepsilon & \text{ whenever } " \\ |3(x - 5)| < \varepsilon & \text{ whenever } " \\ 3|x - 5| < \varepsilon & \text{ whenever } " \\ |x - 5| < \frac{\varepsilon}{3} & \text{ whenever } " \end{aligned}$$

So, choose $\delta = \frac{\varepsilon}{3}$

- Part 2: Show that this δ works:

Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{3}$

$$\begin{aligned} \text{If } 0 < |x - 5| < \delta, \text{ then } |3x - 4 - 11| &= |3x - 15| \\ &= |3(x - 5)| \\ &= 3|x - 5| \\ &< 3 \cdot \delta \\ &= 3 \cdot \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

$\therefore \lim_{x \rightarrow 5} (3x - 4) = 11$

5. Prove that $\lim_{x \rightarrow 2} (x^2 - 3x + 3) = 1$:

Part I: Find a δ that works:

Given an $\varepsilon > 0$, we need a $\delta > 0$ such that

$$\left| (x^2 - 3x + 3) - 1 \right| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta$$

But $\left| (x^2 - 3x + 3) - 1 \right| < \varepsilon$ is equivalent to:

$$|x^2 - 3x + 2| < \varepsilon$$

$$|(x-1)(x-2)| < \varepsilon$$

$$|x-1| \cdot |x-2| < \varepsilon$$

If we can find some number C such that $|x-1| < C$, then we will have that

$$|x-2| < \frac{\varepsilon}{C}, \text{ and we will choose } \delta = \frac{\varepsilon}{C}.$$

Aside: Find C :

A reasonable choice for δ is 1. If we choose $\delta=1$, then we have that:

$$0 < |x-2| < 1$$

$$-1 < x-2 < 1$$

$$0 < x-1 < 2$$

$$|x-1| < 2$$

So, we now have a value for C to be 2. Thus, an alternative to 1 as a choice for δ is $\frac{\varepsilon}{2}$.

Now that we have two values for δ , to ensure that we obtain the desired tolerance value ε , we will choose $\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$.

Part II: Show that our choice for δ works:

Given $\varepsilon > 0$, choose $\delta = \min\left\{1, \frac{\varepsilon}{2}\right\}$.

$$\begin{aligned} \text{If } 0 < |x-2| < \delta, \text{ then } \left| (x^2 - 3x + 3) - 1 \right| &= |x^2 - 3x + 2| \\ &= |(x-1)(x-2)| \\ &= |x-1| \cdot |x-2| \end{aligned}$$

Note: If $\delta=1$, then $0 < |x-2| < 1$

Which implies $-1 < x-2 < 1$

i.e., $0 < x-1 < 2$

or $|x-1| < 2$

But, if $\delta = \frac{\varepsilon}{2}$, then we have that $0 < |x-2| < \frac{\varepsilon}{2}$

$$\begin{aligned} \text{Now we have that } \left| (x^2 - 3x + 3) - 1 \right| &= |x-1| \cdot |x-2| \\ &< 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

We have shown that if $0 < |x-2| < \delta$, then $\left| (x^2 - 3x + 3) - 1 \right| < \varepsilon$

$$\therefore \lim_{x \rightarrow 3} (x^2 - 3x + 3) = 1$$

6. Prove that $\lim_{x \rightarrow 2} (2x^2 - x - 2) = 4$:

Part I: Find a δ that works:

Given an $\varepsilon > 0$, we need a $\delta > 0$ such that

$$\left| (2x^2 - x - 2) - 4 \right| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta$$

$$\text{But } \left| (2x^2 - x - 2) - 4 \right| < \varepsilon \text{ whenever } 0 < |x - 2| < \delta$$

is equivalent to:

$$\left| 2x^2 - x - 6 \right| < \varepsilon \quad \text{whenever } 0 < |x - 2| < \delta$$

$$\left| (2x + 3)(x - 2) \right| < \varepsilon \quad \text{whenever } 0 < |x - 2| < \delta$$

$$\left| 2x + 3 \right| |x - 2| < \varepsilon \quad \text{whenever } 0 < |x - 2| < \delta$$

If we can find some number C such that $|2x + 3| < C$, then we will have that

$$|x - 2| < \varepsilon / C, \text{ and we will choose } \delta = \varepsilon / C.$$

Aside: Find C :

A reasonable choice for δ is 1. If we choose $\delta = 1$, then we have that:

$$0 < |x - 2| < 1$$

$$-1 < x - 2 < 1$$

$$-2 < 2x - 4 < 2$$

$$5 < 2x + 3 < 9$$

$$\left| 2x + 3 \right| < 9$$

So, we now have a value for C to be 9. Thus, an alternative to 1 as a choice for δ is $\varepsilon / 9$.

Now that we have two values for δ , to ensure that we obtain the desired tolerance value ε , we will choose $\delta = \min\{1, \varepsilon / 9\}$.

Part II: Show that our choice for δ works:

Given $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon / 9\}$.

$$\begin{aligned} \text{If } 0 < |x - 2| < \delta, \text{ then } \left| (2x^2 - x - 2) - 4 \right| &= \left| 2x^2 - x - 6 \right| \\ &= \left| (2x + 3)(x - 2) \right| \\ &= \left| 2x + 3 \right| \cdot |x - 2| \end{aligned}$$

Note: If $\delta = 1$, then $0 < |x - 2| < 1$

which implies $-1 < x - 2 < 1$

so $-2 < 2x - 4 < 2$

i.e., $5 < 2x + 3 < 9$

or $\left| 2x + 3 \right| < 9$

But, if $\delta = \varepsilon / 9$, then we have that $0 < |x - 2| < \varepsilon / 9$

Now we have that $\left| (2x^2 - x - 2) - 4 \right| = \left| 2x + 3 \right| \cdot |x - 2|$

$$< 9 \cdot \varepsilon / 9$$

$$= \varepsilon$$

We have shown that if $0 < |x - 2| < \delta$, then $\left| (2x^2 - x - 2) - 4 \right| < \varepsilon$

$$\therefore \lim_{x \rightarrow 2} (2x^2 - x - 2) = 4$$

