Provide both a clear and organized presentation. Show all of your work and give exact values only. A definition of a ring is on the last page of this exam.

1. (15 pts) Prove that if $a = b \bmod 2$ and $a = c \bmod 2$, then $b = c \bmod 2$

$$a = b \bmod 2 \text{ and } a = c \bmod 2 \Rightarrow 2 \mid (a-b) \text{ and } 2 \mid (a-c)$$
$$\Rightarrow a - b = 2k_1 \text{ and } a - c = 2k_2 \text{ for some } k_1, k_2 \in \mathbb{Z}$$

Subtracting the first equation from the second gives us that:

$$b - c = 2k_2 - 2k_1 \text{ for some } k_1, k_2 \in \mathbb{Z}$$
$$\Rightarrow b - c = 2(k_2 - k_1)$$
$$\Rightarrow b - c = 2k_3 \text{ where } k_3 = k_1 + k_2 \in \mathbb{Z}$$
$$\Rightarrow 2 \mid (b-c)$$
$$\Rightarrow b = c \bmod 2$$

2. (25 pts) Determine why each of the following does not form a ring over the indicated operations:

i) $R = \{a \cdot \phi \mid a \in \mathbb{Q} \wedge \phi \text{ is the golden ratio}\}$ with the usual operations of addition and multiplication in $\mathbb{R}$. Recall that $\phi = \dfrac{1+\sqrt{5}}{2}$

$$\begin{aligned}\phi \cdot \phi &= \frac{1+\sqrt{5}}{2} \cdot \frac{1+\sqrt{5}}{2} \\ &= \frac{6+2\sqrt{5}}{4} \\ &= \frac{3+\sqrt{5}}{2} \\ &\neq k \cdot \frac{1+\sqrt{5}}{2} \text{ for any rational number } k\end{aligned}$$

So, we do not have closure for multiplication.

ii) $R = \left\{ \dfrac{1}{n} \middle| n \in \mathbb{N} \right\}$ with the usual operations of addition and multiplication in $\mathbb{Q}$.

$\dfrac{1}{2} + \dfrac{1}{3} = \dfrac{5}{6} \notin R$, so we do not have closure for addition.

iii) $R=\mathbb{R}$ with the usual operation of addition in $\mathbb{R}$ and multiplication is defined by $a\cdot b=1 \quad \forall a,b\in\mathbb{R}$.

$$\begin{aligned} a\cdot\left(b+c\right)&=1 \\ &\neq 1+1 \\ &= a\cdot b+a\cdot c \end{aligned}$$

Consequently, we fail to satisfy our distributive properties.

3. (15 pts) Use an augmented matrix to derive that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & -c\cdot\frac{b}{a}+d & -\frac{c}{a} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{bc}{a\left(ad-bc\right)}+\frac{1}{a} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{bc+ad-bc}{a\left(ad-bc\right)} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

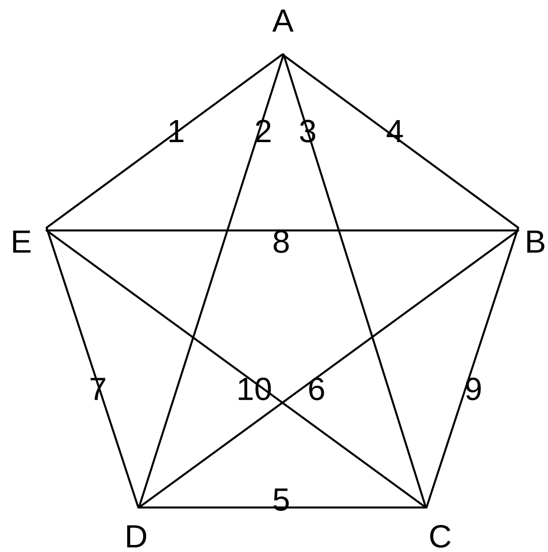
$$\sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Consequently, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

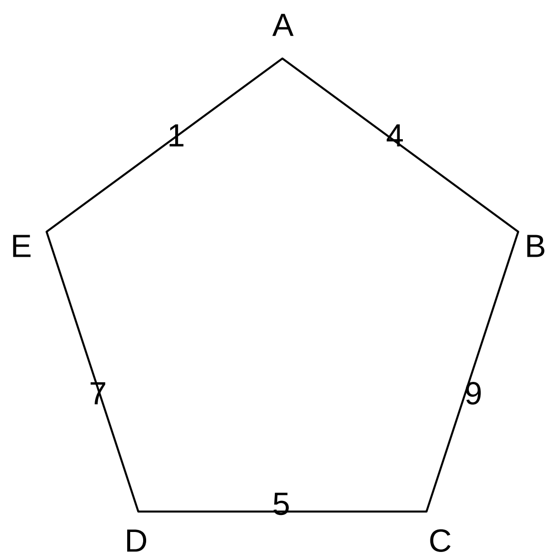
4. (15 pts) Solve the following system of linear equations by either the inverse matrix method or Cramer's rule:

$$\begin{aligned} 2x + y - 3z &= -14 \\ x + 2y + 5z &= 24 \\ 3x - y + 2z &= 14 \end{aligned}$$

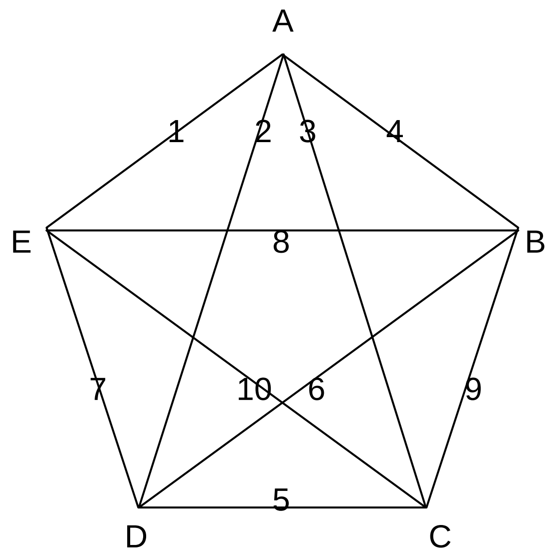
5. (10 pts) Consider the following graph:



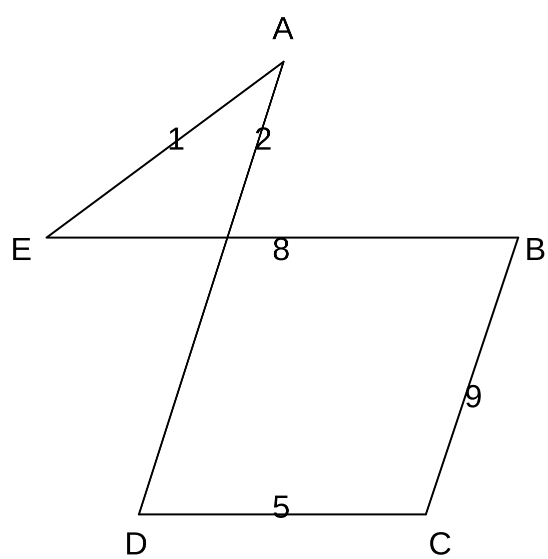
Use the nearest neighbor method to minimize a circuit starting at vertex A. Give its path (i.e., A-E-C-etc.) and its value.



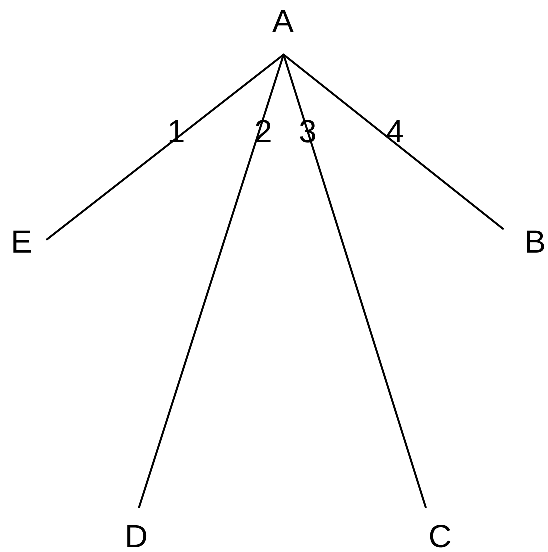
6. (10 pts) Consider the following graph:



Use the cheapest link method to minimize a circuit starting at vertex A. Give its path (i.e., A-E-C-etc.) and its value.



7. (10 pts) Consider the following graph:



Determine the minimal spanning tree for the above graph, draw it below and provide its value.

Def: A ring is a non-empty set R along with two operations $+,\cdot$ satisfying:

$$a+b=b+a \quad \forall a,b \in R$$
$$(a+b)+c=a+(b+c) \quad \forall a,b,c \in R$$
$$\exists e \in R \text{ such that } a+e=e+a=a \quad \forall a \in R$$
$$\forall a \in R, \exists b \in R \text{ such that } a+b=b+a=e$$
$$(a\cdot b)\cdot c=a\cdot(b\cdot c) \quad \forall a,b,c \in R$$
$$a\cdot(b+c)=a\cdot b+a\cdot c \quad \forall a,b,c \in R$$
$$(a+b)\cdot c=a\cdot c+b\cdot c \quad \forall a,b,c \in R$$

Recall that $\mathbb{N}$ = the set of natural numbers
$\mathbb{Q}$ = set of rational numbers
$\mathbb{R}$ = set of real numbers