Provide a clear and organized presentation. Show all of your work, completely simplify your answers, and provide exact values only.

1. My cat *Pythagoras* has amassed a large collection of pets. He has 7 lizards, five mice, and 8 snakes. Determine the probability that if he grabs three of these critters with which to play, they will be of the same type.

$$\binom{7}{3}+\binom{5}{3}+\binom{8}{3}=101$$

2. Meanwhile, my other cat *Theta* is playing with my coins. He has in his possession one of my unfair coins that has a probability of $\frac{2}{3}$ for showing *heads*. What is the probability of tossing this coin three times and having a *heads* showing at least twice?

$$\binom{3}{2}\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{3}+\binom{3}{3}\cdot\left(\frac{2}{3}\right)^3=\frac{20}{27}$$

3. I have been worried that *Pythagoras* has depression related to the rain. For the past 80 days, I have noted which days have experienced rain and which days during which *Pythagoras* has had a good day. Here are my findings:

	good	bad	
rain	20	10	30
no rain	30	20	50
	50	30	80

Does the rain have an effect on *Pythagoras'* mood?

Yes, because $p(g)=\frac{50}{80}=\frac{5}{8}\neq\frac{2}{3}=\frac{20}{30}=p(g|r)$

4. Answer each of the following:

i) Let p and q be natural numbers. If $gcf(p,q)=7$, then determine $lcm(p,q)$

$$lcm(p,q)=\frac{pq}{7}$$

ii) Let m and n be natural numbers. Under what conditions is $gcf\left(m,n\right)=n$?

m is a multiple of n (or n is a factor of m).

5. Let p, q, and r be distinct primes and let a, b, and c be natural numbers for which $a<b<c$.

i) Determine both the $gcf\left(p^b q^c r^a, p^c q^a r^b\right)$ and the $lcm\left(p^b q^c r^a, p^c q^a r^b\right)$

The gcf is $p^b q^a r^a$ and the lcm is $p^c q^c r^b$

ii) If the $lcm\left(a,b\right)=p^8 q^{13}$ and $a=p^8 q^7$, then determine a possible representation of b.

b has not more than 8 factors of p and exactly 13 factors of q.

6. Consider the Fibonacci sequence: $f_1, f_2, f_3, f_4, \ldots$ Next, create a new sequence $d_1, d_2, d_3, d_4, \ldots$ for which $d_n=\frac{f_{n+1}}{f_n}$. Determine the value of $\lim\limits_{n\to\infty} d_n$

d is, of course, simply the golden ratio.

7. In the *Republic of Flatland*, cars are either sedans or coups. In particular, 30% of all cars are sedans and 70% are coups. However, 10% of the sedans are outdated, whereas 20% of all coups are outdated. What is the probability that an arbitrarily chosen car is a sedan if it is known that it is outdated?

$$\frac{30\%\cdot 10\%}{30\%\cdot 10\%+70\%\cdot 20\%}=\frac{3}{17}$$

8. Use the *Euclidean algorithm* to determine $gcf\left(2100,6237\right)$ and then rewrite this gcf as a linear combination of 2100 and 6237.

$$6237=2\cdot 2100+2037$$
$$2100=1\cdot 2037+63$$
$$2037=32\cdot 63+21$$
$$63=3\cdot 21$$

Because 21 is the last non-zero remainder, it is the gcf and

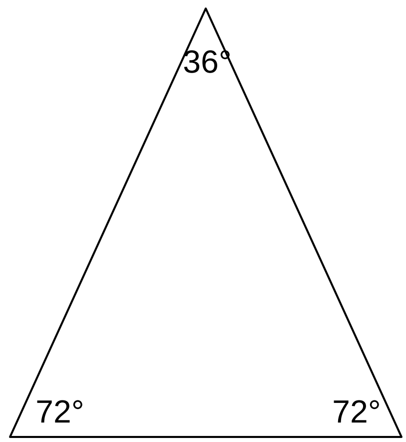
$$21=-98\cdot 2100+33\cdot 6237$$

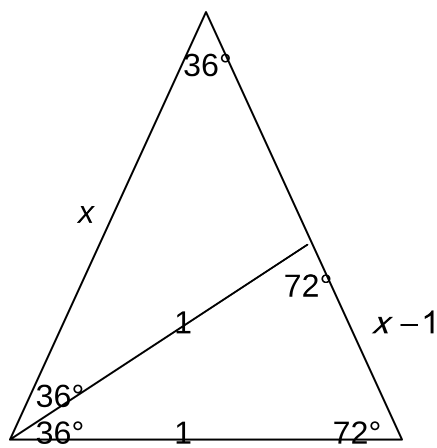
9. Prove that $(A \setminus C) \cap (B \setminus C) \subseteq (A \cup B) \setminus C$

let $x \in (A \setminus C) \cap (B \setminus C)$
$\Rightarrow x \in (A \setminus C)$ and $x \in (B \setminus C)$
$\Rightarrow x \in (A \setminus C)$
$\Rightarrow x \in A$ and $x \notin C$
$\Rightarrow (x \in A$ or $x \in B)$ and $x \notin C$
$\Rightarrow x \in A \cup B$ and $x \notin C$
$\Rightarrow x \in (A \cup B) \setminus C$

$/ \therefore (A \setminus C) \cap (B \setminus C) \subseteq (A \cup B) \setminus C$

10. Show that the following is a golden triangle:





Now the small lower triangle is similar to the original large one. Consequently, sides are proportional. In particular,

$$\begin{aligned}
\frac{x}{1} &= \frac{1}{x-1} \\
x(x-1) &= 1 \\
x^2 - x - 1 &= 0 \\
x &= \frac{1+\sqrt{5}}{2} \\
&= \phi
\end{aligned}$$