<u>Math 15</u>	Exam I	February 14, 2018

Provide both a clear and organized presentation. Completely answer each question, give exact values only, and show all of your work. Only a scientific calculator can be used on this exam. Unless otherwise specified, each question is worth 12 points. The collective symbols \exists ! are banned from this exam.

1. If $A_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{i}\right\} \cup \left(-\frac{1}{i}, 0\right]$, determine each of the following:

i)
$$\bigcup_{i=1}^{n} A_{i}$$

$$= \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}\right\} \cup \left(-1, 0\right]$$

ii)
$$\bigcap_{i=1}^{n} A_{i}$$

$$= \left\{1\right\} \cup \left(-\frac{1}{n}, 0\right]$$

2. (4 pts) How many words (not necessarily meaningful ones) can be formed from using all of the letters in the word *totalitarian*?

$$\frac{12!}{3!3!2!} = 6,652,800$$

- 3. On the distant Island of *Misfit Mathematicians*, people play a version of poker using a strange deck of cards that consists of 6 suits, each containing 8 denominations. In how many ways can one of these islanders draw:
 - i) four cards of the same suit?

$$6 \cdot \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 420$$

ii) seven cards, four of the same denomination, but that last three distinct from each other and distinct from the denomination of the first four?

$$8 \cdot \binom{6}{4} \cdot \frac{42 \cdot 36 \cdot 30}{3!} = 907,200$$

- 4. Let R(x, y) represent the statement: *cat x respects cat y*. Translate each of the following and write your answer in a manner that is as colloquial as possible.
 - i) $\forall x \exists y R(x, y)$ Every cat respects another.

- ii) $\forall y \exists x R(x, y)$ Every cat is respected by another.
- iii) $\exists x \forall y R(x, y)$ Some cat respects all others.
- iv) $\exists y \forall x R(x, y)$ Some cat is respected by all others.
- 5. Let R(x, y) represent the statement: *cat x respects cat y*. Translate each of the following using quantifiers:
 - i) No cat respects itself.

$$\neg \exists x R(x,x) \text{ or } \forall x [\neg R(x,x)]$$

ii) No cat is respected by all others.

$$\neg \exists y \forall x R(x, y) \text{ or } \forall y \exists x [\neg R(x, y)]$$

iii) All cats respect themselves and exactly one other cat.

$$\forall x \Big(R(x,x) \land \exists y \Big(R(x,y) \land (y \neq x) \land \forall z \Big(R(x,z) \rightarrow ((z=y) \lor (z=x)) \Big) \Big) \Big)$$

- 6. Prove that $\sqrt{11} \notin Q$
- 7. Prove that $3^{2n} 2^n$ is divisible by $7 \forall n \in N$
 - i) if $n = 1, 3^{2 \cdot 1} 2^1 = 3^2 2 = 9 2 = 7$ which is clearly divisible by 7.
 - ii) Sp $3^{2k} 2^k$ is divisible by *k* (we will show that $3^{2(k+1)} - 2^{k+1}$ is divisible by *k*)

$$3^{2^{(k+1)}} - 2^{k+1} = 9 \cdot 3^{2^{k}} - 2 \cdot 2^{k}$$

= 9 \cdot 3^{2^{k}} - 9 \cdot 2^{k} + 9 \cdot 2^{k} - 2 \cdot 2^{k}
= 9 \cdot (3^{2^{k}} - 2^{k}) - 7 \cdot 2^{k}
= 9 \cdot 7 \beta - 7 \cdot 2^{k}
= 7 \cdot (9 \beta - 2^{k})
= 7 \cdot q

8. Prove that
$$\sum_{i=1}^{n} 10^{i} = \frac{1}{9} (10^{n+1} - 10)$$
 is divisible by 7 $\forall n \in N$
i) if $n = 1, \sum_{i=1}^{1} 10^{i} = 10 = \frac{1}{9} \cdot 90 = \frac{1}{9} \cdot (10^{2} - 10)$

ii) Sp
$$\sum_{i=1}^{k} 10^{i} = \frac{1}{9} (10^{k+1} - 10)$$

(we will show that $\sum_{i=1}^{k+1} 10^{i} = \frac{1}{9} (10^{k+2} - 10)$)

$$\sum_{i=1}^{k+1} 10^{i} = \sum_{i=1}^{k} 10^{i} + 10^{k+1}$$
$$= \frac{1}{9} (10^{k+1} - 10) + \frac{1}{9} \cdot 9 \cdot 10^{k+1}$$
$$= \frac{1}{9} (10^{k+1} - 10 + 9 \cdot 10^{k+1})$$
$$= \frac{1}{9} (10 \cdot 10^{k+1} - 10)$$
$$= \frac{1}{9} (10^{k+2} - 10)$$

9. Designate letters to represent the propositions within the following argument, then list both the steps and reasons for those steps. Establish the validity of the argument using our rules of inference.

If my cat *Pythagoras* starts a fire in my house, it will both cause devastation to my household and embarrassment for my family. My cat *Pythagoras* has not caused embarrassment for my family. If my cat does not start a fire in my house, I will be able to sleep peacefully. Therefore, I sleep peacefully.

Let *f* represent *Pythagoras* starts a fire in my house, *d* represent devastation will occur in my house, *e* represent there will be embarrassment for my family, *p* represent I will sleep peacefully.

- 1. *¬e*
- $\begin{array}{ll} 2. & \neg e \lor \neg d \\ 3. & \neg d \lor \neg e \end{array}$
- $3. \quad \neg d \lor \neg e$ $4. \quad \neg (d \land e)$

- 1. given
- 2. disjunctive amplification
- 3. commutativity of \vee
- 4. DeMorgan's law

5.
$$f \rightarrow (d \wedge e)$$

6.
$$\neg f$$

7 $\neg f$

$$7. \qquad \neg f \to p$$

8. :. p

given 5.

modus tollens given 6.

7.

modus ponens 8.