Math 12 Solving a System of Three Equations in Three Unknowns Fall, 2018

In our investigation of techniques that will allow us to solve a system of three equations in three unknowns, we will discover several techniques. Namely, we will investigate the *Elimination Method*, *Gauss-Jordan Elimination (used on an augmented matrix)*, using a single matrix equation, and *Cramer's Rule*.

Consider the following system:

$$\begin{aligned} 2x - 2y - 4z &= -6 \\ 2x + y - z &= -3 \\ x + y + 2z &= 5 \end{aligned}$$

I. The Elimination Method:

We will use the first two equations to eliminate the variable *x* and arrive at a single equation in only *y* and *z*. Multiplying the second equation by –1 and adding, we have:

$$\begin{aligned} 2x - 2y - 4z &= -6 \\ -2x - y + z &= 3 \\ \hline -3y - 3z &= -3 \qquad (1) \end{aligned}$$

We will now use the last two equations to eliminate the variable *x* and arrive at a second equation in only *y* and *z*. Multiplying the second equation by –2 and adding, we have:

$$\begin{aligned} 2x + y - z &= -3 \\ -2x - 2y - 4z &= -10 \\ \hline -y - 5z &= -13 \qquad (2) \end{aligned}$$

Using the equations (1) and (2), we will now consider an easier system of two equations in two unknowns:

$$\begin{aligned} -3y - 3z &= -3 \\ -y - 5z &= -13 \end{aligned}$$

Multiplying both sides of the first equation by $-\frac{1}{3}$, then adding, we can eliminate the variable *y* from the system and arrive at an equation solely in *z*:

$$\begin{aligned} y + z &= 1 \\ -y - 5z &= -13 \\ \hline -4z &= -12 \\ z &= 3 \qquad (3.1) \end{aligned}$$

We will now find the value for the variable *y* by using our value for *z* and substituting back into equation (2):

$$
\begin{aligned}
-y - 5\cdot 3 &= -13 \\
-y - 15 &= -13 \\
-y &= 2 \\
y &= -2 \qquad (3.2)
\end{aligned}
$$

Finally, we will now use our determined values for both *y* and *z*, substitute them into one of the original three equations (we will use the third, for simplicity), and determine the value for the one remaining variable *x*:

$$
\begin{aligned}
x + (-2) + 2\cdot 3 &= 5 \\
x + 4 &= 5 \\
x &= 1 \qquad (3.3)
\end{aligned}
$$

From the equations (3.1), (3.2), and (3.3), we now have the ordered triple solution: $(1, -2, 3)$

II. Gauss-Jordan Elimination on an Augmented Matrix:

Rewriting our original system as an augmented matrix, we will apply a sequence of elementary row operations to rewrite that augmented matrix in reduced row-echelon form:

$$
\left[\begin{array}{ccc|c} 2 & -2 & -4 & -6 \\ 2 & 1 & -1 & -3 \\ 1 & 1 & 2 & 5 \end{array}\right]
$$

$$
\frac{1}{2}R_1 \to R_1 \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 2 & 1 & -1 & -3 \\ 1 & 1 & 2 & 5 \end{array}\right]
$$

$$
\begin{array}{r} -2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 3 \\ 0 & 2 & 4 & 8 \end{array}\right]
$$

$$
\frac{1}{3}R_2 \to R_2 \left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \end{array}\right]
$$

$$
\begin{array}{r} R_2 + R_1 \to R_1 \\ -2R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \end{array}\right]
$$

$$\frac{1}{2}R_3 \to R_3 \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{r} R_3 + R_1 \to R_1 \\ -R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

translating back from augmented matrix form to equation form, this gives us an equivalent system of three equations in the same three unknowns, specifically:

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3 \end{aligned}$$

Again, we have a solution that is the ordered triple $(1, -2, 3)$

Now let us consider two additional techniques for solving a system of three equations in three unknowns:

III. Using a Single Matrix Equation:

We can rewrite the original system of three equations in three unknowns as a single matrix equation where A represents the matrix of coefficients of our three variables *x*, *y*, and *z*. The matrix $\overline{x}$, will be a 3×1 matrix (or equivalently a column vector) consisting of these three variables and $\overline{b}$ will be a 3×1 matrix containing the constants that reside on the right-hand sides of the equations in the original system.

So our system can be generically rewritten in the form:

$$A\overline{x} = \overline{b}$$

Of course, our solution to this equation is:

$$\overline{x} = A^{-1}\overline{b}$$

With our system, we can rewrite as a single matrix equation:

$$\begin{bmatrix} 2 & -2 & -4 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 5 \end{bmatrix}$$

whose solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ -3 \\ 5 \end{bmatrix} \qquad (*)$$

Unfortunately, we cannot multiply on the right-hand side of this equation without first taking a break and finding A^{-1}.

Aside:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & -4 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 2 & -2 & -4 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & -1 & -5 & 0 & 1 & -2 \\ 0 & -4 & -8 & 1 & 0 & -2 \end{array}\right]$$

$$-R_2 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 5 & 0 & -1 & 2 \\ 0 & -4 & -8 & 1 & 0 & -2 \end{array}\right]$$

$$\begin{array}{r} -R_2 + R_1 \to R_1 \\ 4R_2 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & -1 \\ 0 & 1 & 5 & 0 & -1 & 2 \\ 0 & 0 & 12 & 1 & -4 & 6 \end{array}\right]$$

$$\frac{1}{12}R_3 \to R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & -1 \\ 0 & 1 & 5 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1/12 & -1/3 & 1/2 \end{array}\right]$$

$$\begin{array}{r} 3R_3 + R_1 \to R_1 \\ -5R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 0 & 1/2 \\ 0 & 1 & 0 & -5/12 & 2/3 & -1/2 \\ 0 & 0 & 1 & 1/12 & -1/3 & 1/2 \end{array}\right]$$

Now that we have found that $A^{-1} = \begin{bmatrix} 1/4 & 0 & 1/2 \\ -5/12 & 2/3 & -1/2 \\ 1/12 & -1/3 & 1/2 \end{bmatrix}$, we can multiply on the right-hand side of the equation (*):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ -3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/4 & 0 & 1/2 \\ -5/12 & 2/3 & -1/2 \\ 1/12 & -1/3 & 1/2 \end{bmatrix} \begin{bmatrix} -6 \\ -3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\cdot(-6)+0\cdot(-3)+\frac{1}{2}\cdot 5 \\ -\frac{5}{12}\cdot(-6)+\frac{2}{3}\cdot(-3)-\frac{1}{2}\cdot 5 \\ \frac{1}{12}(-6)-\frac{1}{3}\cdot(-3)+\frac{1}{2}\cdot 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Yet again, we have a solution that is the ordered triple $(1, -2, 3)$

For our final attempt at solving this system of three equations in three unknowns, we will employ what is referred to as *Cramer's Rule*, which employs the use of determinants:

IV. Cramer's Rule:

Let us first compute D, the determinant of the coefficient matrix for the system of equations:

$$D = \begin{vmatrix} 2 & -2 & -4 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - (-2)\cdot\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + (-4)\cdot\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 2\cdot 3 - (-2)\cdot 5 + (-4)\cdot 1$$

$$= 6 + 10 - 4$$

$$= 12$$

Replacing the first column (the column consisting of the coefficients for the variable x in each of our three equations) of the determinant D with the constants on the right-hand side of the equal signs in each of the original equations of our system, we can compute D_x:

$$D_x = \begin{vmatrix} -6 & -2 & -4 \\ -3 & 1 & -1 \\ 5 & 1 & 2 \end{vmatrix} = -6 \cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - (-2) \cdot \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} + (-4) \cdot \begin{vmatrix} -3 & 1 \\ 5 & 1 \end{vmatrix}$$

$$\begin{aligned} &= -6 \cdot 3 - (-2) \cdot (-1) + (-4) \cdot (-8) \\ &= -18 - 2 + 32 \\ &= 12 \end{aligned}$$

Repeating this process on the second and third columns, we arrive at:

$$D_y = \begin{vmatrix} 2 & -6 & -4 \\ 2 & -3 & -1 \\ 1 & 5 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} -3 & -1 \\ 5 & 2 \end{vmatrix} - (-6) \cdot \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + (-4) \cdot \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix}$$

$$\begin{aligned} &= 2 \cdot (-1) - (-6) \cdot 5 + (-4) \cdot 13 \\ &= -2 + 30 - 52 \\ &= -24 \end{aligned}$$

and

$$D_z = \begin{vmatrix} 2 & -2 & -6 \\ 2 & 1 & -3 \\ 1 & 1 & 5 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -3 \\ 1 & 5 \end{vmatrix} - (-2) \cdot \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} + (-6) \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 2 \cdot 8 - (-2) \cdot 13 + (-6) \cdot 1 \\ &= 16 + 26 - 6 \\ &= 36 \end{aligned}$$

Finally, we have that $x = \frac{D_x}{D} = \frac{12}{12} = 1$,

$y = \frac{D_y}{D} = \frac{-24}{12} = -2$, and

$z = \frac{D_z}{D} = \frac{36}{12} = 3$

And one last time, we arrive at the solution that is the ordered triple $(1, -2, 3)$